

On Characters of Inductive Limits of Symmetric Groups

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Abstract

In the paper we completely describe characters (central positive-definite functions) of simple locally finite groups that can be represented as inductive limits of (products of) symmetric groups under block diagonal embeddings. Each such group G defines an infinite graph (Bratteli diagram) that encodes the embedding scheme. The group G acts on the space X of infinite paths of the associated Bratteli diagram by changing initial edges of paths. Assuming the finiteness of the set of ergodic measures for the system (X, G) , we establish that each indecomposable character $\chi : G \rightarrow \mathbb{C}$ is uniquely defined by the formula $\chi(g) = \mu_1(Fix(g))^{\alpha_1} \cdots \mu_k(Fix(g))^{\alpha_k}$, where μ_1, \dots, μ_k are G -ergodic measures, $Fix(g) = \{x \in X : gx = x\}$, and $\alpha_1, \dots, \alpha_k \in \{0, 1, \dots, \infty\}$. We illustrate our results on the group of rational permutations of the unit interval.

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1 Introduction

The representation theory of groups, originally motivated by the study of finite groups, eventually turned into a powerful framework for formulating and solving problems from numerous areas of mathematics and theoretical physics. A later development of the representation theory of infinite groups connected the group theory to such mathematical areas as the free probability theory, random matrices, algebraic geometry, and many others. This was mainly established via the group $S(\infty)$ of all finitary permutations of the set of natural numbers, see the discussion and references in the monograph [K].

The classification of characters for the group $S(\infty)$ was obtained by Thoma [Th] in 1964. Later Kerov and Vershik [VK1, VK2] showed that indecomposable

characters of $S(\infty)$ can be found as weak limits of characters of groups $S(n)$. They called this construction the *ergodic approach or the asymptotic theory of characters*. We note that the application of the ergodic approach in [VK2] provided a very neat interpretation of Thoma's parameters as frequencies of rows and columns in the Young diagram. We also mention a recent preprint [GK], where the authors give a purely operator algebra proof of the classification of characters for $S(\infty)$, and the paper [Ok] by Okounkov, who used the Olshanski semigroup approach to give another proof of Thoma's theorem.

In the present paper we are interested in the theory of characters for groups that can be represented as inductive limits of symmetric groups under diagonal embeddings. One of the simplest examples of such a group can be given as follows. Consider the sequence of sets $X_n = \{0, \dots, 2^n - 1\}$ and the symmetric groups $S(X_n)$. The set X_{n+1} can be represented as a disjoint union of two copies of X_n . Then each element $s \in S(X_n)$ can be embedded into $S(X_{n+1})$ by making it act on each copy of X_n as s . Define the group $S(2^\infty)$ as the inductive limit of groups $S(X_n)$ under this embedding scheme. The characters of $S(2^\infty)$ were completely described by the first-named author in [D]. We would like to mention that the technique used in [D], which is further developed in the present paper, is significantly different from the ergodic approach of Vershik and Kerov and is solely based on the study of actions of $S(2^\infty)$ and *not* on the analysis of weak limits of characters.

Characters of groups similar to $S(2^\infty)$ were also studied by Leinen and Puglisi in [LePu], who applied a version of the ergodic approach to describe characters for inductive limits of alternating groups $A(X_n)$ with the embedding scheme allowing the set X_{n+1} to be covered by multiple copies of X_n .

The group $S(2^\infty)$ can be represented as the group of homeomorphisms of $\{0, 1\}^\mathbb{N}$ preserving the tails of sequences or as the group permuting initial segments of infinite paths of the Bratteli diagram corresponding to the 2-odometer. Many groups covered in [LePu] can be also seen as groups of permutations of Bratteli diagrams (corresponding to odometers).

Following this observation, we will consider groups acting on initial segments of infinite paths of *general* Bratteli diagram, see Definition 2.8. We will refer to these groups as *full groups*¹ of Bratteli diagrams. Algebraically, full groups are inductive limits of products of symmetric groups under block diagonal embeddings (similar to those used in $S(2^\infty)$), see Section 2.2 for the details. We notice that full groups are countable and locally finite. The algebraic structure of full groups and groups isomorphic to them were studied in [KrSu, LS, LN, M, BM, Z]. A discussion of the connection between full groups and the classification of general locally finite groups can be found in [LS, LN].

Consider the full group G of a Bratteli diagram B . Then the group G acts on the space X of infinite paths of B . The set X can be endowed with a topology that turns it into a Cantor set. We will assume that the group G is simple, which implies that the dynamical system (X, G) is minimal (every G -orbit is dense). Fix an indecomposable character $\chi : G \rightarrow \mathbb{C}$, i.e. a central positive-

¹These groups are also called AF (approximately finite) full groups, see [M].

definite function that cannot be represented as a convex combination of similar functions. The main goal of the paper is to find a closed-form expression for χ . Our approach to this problem is based on the study of action of the group G on the path-space X . In our proofs, we use some ideas of [D].

Applying the GNS-construction to the character χ , one can find a finite-type factor representation² π of the group G in a Hilbert space H and a vector $\xi \in H$ such that $\chi(g) = (\pi(g)\xi, \xi)$, see the details in Section 2.3. Denote by \mathcal{M}_π the W^* -algebra generated by the unitary operators $\pi(G)$. Observe that \mathcal{M}_π has a unique normalized trace tr given by $tr(T) = (T\xi, \xi)$.

The first main result of our paper is the proof of the following theorem establishing that the value of the character χ at the element $g \in G$ depends only on the set $supp(g) = \{x \in X : g(x) \neq x\}$. For a clopen set A , denote by $G(A)$ the subgroup of all elements $g \in G$ supported by the set A . Let P^A stand for the projection on the subspace $\{h \in H : \pi(g)h = h \text{ for all } g \in G(A)\}$.

Theorem 1.1. *Let G be the simple full group of a Bratteli diagram. Then for every $g \in G$, we have that $P^{supp(g)} \in \mathcal{M}_\pi$ and*

$$\chi(g) = tr(P^{supp(g)}).$$

Observe that groups $G(A)$ play a similar role in G as the finite symmetric groups $S(n)$ do in $S(\infty)$.

Assume that the dynamical system (X, G) has only a finitely many of ergodic measures μ_1, \dots, μ_k . Under this assumption, we will establish the following result.

Theorem 1.2. *For each indecomposable character χ of the simple full group G there exist parameters $\alpha_1, \dots, \alpha_k \in \{0, 1, \dots, \infty\}$ such that*

$$\chi(g) = \mu_1(Fix(g))^{\alpha_1} \cdots \mu_k(Fix(g))^{\alpha_k}$$

for every $g \in G$. Here $Fix(g) = \{x \in X : g(x) = x\}$.

If $\alpha_i = \infty$ for some i , then the character χ is regular, i.e. $\chi(1) = 1$ and $\chi(g) = 0$ for all $g \neq 1$. Theorem 1.2 shows that the description of characters for each given full group is reduced to the description of ergodic measures on the corresponding Bratteli diagram. We mention the following papers [BKMS1, BKMS2, FFT, Mel, PV] devoted to the study of ergodic measures on Bratteli diagrams.

We observe that the characters of the group $S(\infty)$ that take only non-negative values can be also obtained as a measure of fixed points from certain actions of $S(\infty)$, see [VK1]. The description of characters in terms of measures of fixed points also appears in [D], [GoPe], and [LePu]³, see also [Gr, Chapter 9] for characters of certain finitely generated groups.

²The term “finite-type” means that the von Neumann algebra generated by the representation of the group G is of finite type, i.e. either of type II_1 or I_n with $n < \infty$, see the details of this classifications in [Ta, Chapter 5].

³Formally, the characters in [LePu, Theorem 3.2] are given as functions that depend on the cardinality of the set of fixed points. However, these functions can be interpreted as powers of the measure of fixed points.

Theorem 1.2 classifies characters for simple full groups and, as a result, all II_1 -factor representations of G up to quasi-equivalence. Observe that there are Bratteli diagrams with non-simple full groups. Such groups might have characters taking complex values, which cannot be exclusively determined by invariant measures. However, we think that characters of arbitrary full groups can be described as a product of characters “built by measures” and some homomorphisms into the unit circle, cf. [V2]. Denote by $D(G)$ the commutator subgroup of G . Notice that $D(G)$ is a simple group if the dynamical system (X, G) is minimal, see [M] or [LS].

Conjecture. *Let G be the full group (not necessarily simple) of a Bratteli diagram. Then every indecomposable character χ of the group G has the form*

$$\chi(g) = \rho([g]) \cdot \mu_1(\text{Fix}(g))^{\alpha_1} \cdots \mu_k(\text{Fix}(g))^{\alpha_k} \text{ for } g \in G,$$

where $\alpha_1, \dots, \alpha_k \in \{0, 1, \dots, \infty\}$, μ_1, \dots, μ_k are some ergodic G -invariant measures, $[g]$ is the image of g in the (Abelian) quotient group $H = G/D(G)$, and ρ is a homomorphism from the group H into the unit circle $\{z \in \mathbb{C} : |z| = 1\}$.

The structure of the paper is the following. In Section 2, we fix the notations and introduce the main definitions of the paper. Subsection 2.1 is devoted to Bratteli diagrams. Here we give a formal definition of Bratteli diagrams and their full groups. We prove that the full group is simple if and only if the Bratteli diagram can be telescoped to have an even number of edges between any two vertices. In Subsection 2.3 we give the definition of a character and recall the essence of the GNS construction.

Section 3 is mostly technical and contains two lemmas that allow us to move elements of the unitary group $\pi(G)$ to the center of the algebra \mathcal{M}_π by conjugation. This technique was also extensively used in the representation theory of $S(\infty)$, see, for example, [Ok]. We also introduce a topology (Definition 3.4) on the group G similar to the *uniform topology* studied in ergodic theory [H]. We will use this topology to study continuous characters and representations of non-simple full groups.

In Section 4, we study orthogonal projections P^A on the subspace of vectors fixed under the group $\pi(G(A))$. We show that when A is a clopen set, the projection P^A can be obtained as a weak limit of involutions $\{\pi(s_n)\}$ with $\text{supp}(s_n) = A$. Furthermore, we show that the projections $\{P^A\}$ form an Abelian semigroup, i.e. $P^A P^B = P^{A \cup B}$ for any clopen sets A and B . We then prove Theorem 1.1 for simple full groups. We also establish this result for arbitrary full groups under the assumption that the character or the representation is continuous with respect to the uniform topology.

In Section 5 we consider full groups that have only a finitely many of ergodic measures μ_1, \dots, μ_k . We introduce the function $\varphi(\mu_1(A), \dots, \mu_k(A)) = \text{tr}(P^{X \setminus A})$, $A \subset X$, defined on a dense subset of $[0, 1]^k$. We show that the domain of φ can be extended to the entire set $[0, 1]^k$ and that the function is multiplicative and continuous. Note that the multiplicativity of φ can be

seen as a version of multiplicativity of characters on $S(\infty)$, see, for example, [Ok]. Solving Cauchy's functional equation for the function φ , we get that $\varphi(t_1, \dots, t_k) = t_1^{\alpha_1} \cdots t_k^{\alpha_k}$ for some non-negative real numbers $\alpha_1, \dots, \alpha_k$. We then show that these parameters are, in fact, integers, which yields the proof of Theorem 1.2. This description is also valid for arbitrary full groups provided that the character is continuous.

As a corollary of Theorem 1.2 we get that any finite-type factor representation of a simple full group (corresponding to a non-regular character) is automatically continuous with respect to the uniform topology. This indicates that *simple* full groups of Bratteli diagrams have many features of “big” groups such as groups of measure-preserving automorphisms or homeomorphisms of a Cantor set, which possess a stronger automatic continuity property [KeRo, KiT, R, RS].

Theorem 1.2 can be also used to describe characters of full groups of Cantor minimal systems, see Corollary 5.9. This shows that the property of unique ergodicity of minimal systems can be completely reformulated on the language of indecomposable characters.

In Section 6, we build a II_1 -factor representation for each character from Theorem 1.2, whereby we describe all II_1 -factor representations of simple full groups up to quasi-equivalence.

In Section 7 we consider the group of all rational permutations of the unit interval. We apply Theorem 1.2 to get a new proof of the classification of characters for this group, which was originally established by Goryachko and Petrov [Go, GoPe]. We mention that the proof in [GoPe] is based on the asymptotic theory of characters.

2 Preliminaries

In the section we give the definitions of Bratteli diagrams, full groups, characters, and related dynamical notions.

2.1 Bratteli diagrams and the associated full groups

Bratteli diagrams have been studied in many papers on Cantor dynamics and operator algebras. Since in the present paper we are mainly interested in the dynamical nature of Bratteli diagrams, our main references to the theory of such diagrams will be [HPS] and [GPS1].

Definition 2.1. A *Bratteli diagram* is an infinite graph $B = (V, E)$ such that the vertex set $V = \bigcup_{i \geq 0} V_i$ and the edge set $E = \bigcup_{i \geq 1} E_i$ are partitioned into disjoint subsets V_i and E_i such that

- (i) $V_0 = \{v_0\}$ is a single point;
- (ii) V_i and E_i are finite sets;
- (iii) there exist a range map r and a source map s from E to V such that $r(E_i) = V_i$, $s(E_i) = V_{i-1}$, and $s^{-1}(v) \neq \emptyset$, $r^{-1}(v') \neq \emptyset$ for all $v \in V$ and $v' \in V \setminus V_0$.

A graphical example of a Bratteli diagram is given in Section 7. The pair (V_i, E_i) is called the i -th level of the diagram B . A finite or infinite sequence of edges $(e_i : e_i \in E_i)$ such that $r(e_i) = s(e_{i+1})$ is called a *finite* or *infinite path*, respectively. For a Bratteli diagram B , we denote by X_B the set of infinite paths starting at the root vertex v_0 . We will always omit the index B in the notation X_B as the diagram will be always clear from the context. We endow the set X with the topology generated by cylinder sets $U(e_1, \dots, e_n) = \{x \in X : x_i = e_i, i = 1, \dots, n\}$, where (e_1, \dots, e_n) is a finite path from B . Then X is a 0-dimensional compact metric space with respect to this topology. Observe that the space X might have isolated points, but we will assume that the space is perfect, i.e. X is a Cantor set.

Denote by $h_v^{(n)}$, $v \in V_n$, the number of finite paths connecting the root vertex v_0 and the vertex v . Given a vertex $v \in V_n$, enumerate the cylinder sets $U(e_1, \dots, e_n)$ with $r(e_n) = v$ by $\{C_{v,0}^{(n)}, \dots, C_{v,h_v^{(n)}-1}^{(n)}\}$. Then

$$\Xi_n = \{C_{v,i}^{(n)} : v \in V_n, i = 0, \dots, h_v^{(n)} - 1\} \quad (1)$$

is a clopen partition of X .

Notice that if A is a clopen subset of X_B , then we can always find a level $n \geq 1$ such that the set A is a disjoint union of cylinder sets $U(e_1, \dots, e_n)$ of depth n . We will say then that the set A is *compatible with the partition* Ξ_n .

For multiple examples of diagrams as well as their diagrammatic representations we refer the reader to the papers [BKMS1], [BKMS2], [HPS], and [GPS1].

Fix a level $n \geq 0$. For each pair of vertices $(w, v) \in V_{n+1} \times V_n$, denote by $f_{w,v}^{(n)}$ the number of edges connecting them.

Definition 2.2. The $|V_{n+1}| \times |V_n|$ -matrix $F_n = (f_{w,v}^{(n)})$ ($v \in V_n, w \in V_{n+1}$) is called an *incidence matrix* of the diagram.

Remark 2.3. Observe that F_0 is a vector and $h_v^{(n)}$ coincides with the v -th coordinate of the vector $F_{n-1} \cdots F_0$.

Definition 2.4. A Bratteli diagram $B = (V, E)$ is called *simple* if for any level V_n there is a level V_m , $m > n$, such that every pair of vertices $(v, w) \in V_n \times V_m$ is connected by a finite path.

Definition 2.5. We say that two infinite paths $x = (x_n)_{n \geq 1}$ and $y = (y_n)_{n \geq 1}$ of a Bratteli diagram B are *cofinal* or *tail-equivalent* if $x_n = y_n$ for all n large enough. The equivalence relation generated by cofinal paths is called the *cofinal* or *tail* equivalence relation.

Definition 2.6. Let X be a compact space and H a group acting by homeomorphisms of X . We say that the H -action is *minimal* if the H -orbit of every point $x \in X$, $\text{Orb}_H(x) = \{h(x) : h \in H\}$, is dense in X .

Fix a Bratteli diagram $B = (V, E)$ and a level $n \geq 1$. Denote by G_n the group of all homeomorphisms of $X := X_B$ that changes only the first n -edges of each infinite path. Namely, a homeomorphism g belongs to G_n if and only

if for any finite path (e_1, \dots, e_n) there exists a finite path (e'_1, \dots, e'_n) with $r(e'_n) = r(e_n)$ such that

$$g(e_1, \dots, e_n, e_{n+1}, e_{n+2}, \dots) = (e'_1, \dots, e'_n, e_{n+1}, e_{n+2}, \dots)$$

for each infinite extension $(e_1, \dots, e_n, e_{n+1}, e_{n+2}, \dots)$ of the path (e_1, \dots, e_n) . Observe that $G_n \subset G_{n+1}$. Set $G = \bigcup_{n \geq 1} G_n$.

Remark 2.7. (1) G is a locally finite countable group.

(2) Given $x \in X$, the G -orbit of x consists of all infinite paths cofinal to x .

(3) The simplicity of the diagram is equivalent to the minimality of the G -action on X .

(4) Given a level $n \geq 1$ and a vertex $v \in V_n$, denote by $G_n^{(v)}$ the subgroup of G_n such that $g \in G_n^{(v)}$ if $g(x) = x$ for all $x \in X$ with $r(x_n) \neq v$. In other words, the elements of $G_n^{(v)}$ permutes only the infinite paths going through the vertex v . Then

$$G_n = \prod_{v \in V_n} G_n^{(v)}.$$

Definition 2.8. We will call the group G the *full group of the Bratteli diagram* B . The symbol G will be reserved exclusively for the full groups of Bratteli diagrams.

For various (other) notions of full groups in the context of Cantor dynamics, we refer the reader to the papers [GPS2] and [BM]. The main property of the group G , which also holds in other full groups (cf. [GPS2]), is the ability to “glue” elements of G . Namely, if $\{A_1, \dots, A_n\}$ is a clopen partition of X and $\{g_1, \dots, g_n\}$ are elements of G such that $\{g_1(A_1), \dots, g_n(A_n)\}$ is also a clopen partition of X , then the homeomorphism g of X defined by $g(x) = g_i(x)$ (for $x \in A_i$, $i = 1, \dots, n$) also belongs to G .

Definition 2.9. We say that two clopen sets A and B are *G -equivalent* if there is $g \in G$ with $g(A) = B$.

Remark 2.10. (1) Since the group G is amenable (locally finite), the dynamical system (X, G) has at least one G -invariant probability measure. Hence if A and B are G -equivalent, then $\mu(A) = \mu(B)$ for all G -invariant probability measures. Observe that, in general, the converse statement is not true.

(2) Let A be a clopen set such that A can be written as $A = A_1 \sqcup A_2$ with A_1 and A_2 being G -equivalent. Choose $n \geq 1$ so that all sets A_1 , A_2 , and A are compatible with the partition Ξ_n and $g \in G_n$. Then for every vertex $v \in V_n$, there is an even number of finite paths from the root vertex v_0 to v lying completely in the set A .

The following result gives a dynamical criterion for a group G to be simple. We note that a special case of this result, when the diagram B has exactly one vertex at every level, can be derived from Theorem 1 of [KrSu], cf. [M, Corollary 4.10]. Denote by $D(G)$ the commutator subgroup of G , i.e. a subgroup generated by all elements of the form $[g_1, g_2] := g_1 g_2 g_1^{-1} g_2^{-1}$, $g_1, g_2 \in G$. Clearly, $D(G)$ is a normal subgroup of G .

Proposition 2.11. (1) The group $D(G)$ is simple if and only if the associated Bratteli diagram is simple. This is also equivalent to the minimality of the dynamical system (X, G) .

(2) G is simple, i.e. $D(G) = G$, if and only if the diagram is simple and every clopen set $A \subset X$ can be represented as $A = A_1 \sqcup A_2$ with A_1 and A_2 being G -equivalent.

Proof. (1) The first statement was proved in [M, Lemma 3.4], see also [LS, Theorem 2].

(2-i) First of all assume that every clopen set can be decomposed as a union of G -equivalent clopen sets. Fix $g \in G$. Denote by B_k the set of points having g -period exactly k , i.e.

$$B_k = \{x \in X : g^k(x) = x \text{ and } g^l(x) \neq x \text{ for all } 1 \leq l \leq k-1\}.$$

Denote by K the set of all integers k with $B_k \neq \emptyset$. Note that K is a finite set. Choose a clopen set $B_k^{(0)}$ with $g^l(B_k^{(0)}) \cap B_k^{(0)} = \emptyset$ for all $l = 1, \dots, k-1$, and $B_k^{(0)} \cup g(B_k^{(0)}) \cup \dots \cup g^{k-1}(B_k^{(0)}) = B_k$.

Find G -equivalent clopen sets $A_k^{(1)}$ and $A_k^{(2)}$ such that $B_k^{(0)} = A_k^{(1)} \sqcup A_k^{(2)}$. Choose $h_k \in G$ such that $h_k|(X \setminus B_k^{(0)}) = id$, $h_k(A_k^{(1)}) = A_k^{(2)}$, and $h_k^2 = id$. Define a homeomorphism $h \in G$ as follows

$$h(x) = \begin{cases} g^l h_k g^{-l}(x) & \text{for } x \in g^l(B_k^{(0)}), 0 \leq l \leq k-1, k \in K; \\ x & \text{elsewhere.} \end{cases}$$

Set

$$C_i = \bigcup_{k \in K} \bigcup_{l=0}^{k-1} g^l(A_k^{(i)}) \text{ for } i = 1, 2.$$

Define $g_i \in G$ as g on C_i and as id elsewhere. Now, it is a routine to check that $hg_1h^{-1} = g_2$ and $g = g_1g_2 = g_1hg_1h^{-1}$. Noting that g_1 is isomorphic to g_1^{-1} , we conclude that $g \in D(G)$. Therefore, $D(G) = G$.

(2-ii) Now assume that G is a simple group, i.e. $G = D(G)$. Fix a clopen set A . Without loss of generality we may assume that $A = U(e_1, \dots, e_n)$ is a cylinder set and there exists a path $(e'_1, \dots, e'_n) \neq (e_1, \dots, e_n)$ between v_0 and $v = r(e_n)$. To prove the proposition it is enough to show that there exists a level $m > n$ such that for any vertex $w \in V_m$ the cardinality of the set $C_{v,w}$ of finite paths between v and w is even. Then dividing $C_{v,w}$ into subsets $C_{v,w}^{(1)}$ and $C_{v,w}^{(2)}$ of equal cardinalities and setting A_i , $i = 1, 2$, to be the union of cylinders $U(e_1, \dots, e_m)$ with $(e_{n+1}, \dots, e_m) \in C_{v,w}^{(i)}$, we obtain the desired partition.

Find $m > n$ so that every element $g \in G_n$ can be written as a product of commutators $[f, h]$ with $f, h \in G_m$. In other words, the group G_n can be seen as a subgroup of the alternating group $D(G_m)$. Assume that for some $w \in V_m$ the cardinality of $C_{v,w}$ is odd. Let $g \in G_n^{(v)}$ be the transposition $(e_1, \dots, e_n) \leftrightarrow (e'_1, \dots, e'_n)$. Then, the embedding of g into $G_m^{(w)}$ is a product of an odd number of transpositions. Hence, g is an odd permutation in the group $G_m^{(w)}$, i.e. $g \notin D(G_m)$, which is a contradiction. \square

Definition 2.12. Let $B = (V, E)$ be a Bratteli diagram. Fix a sequence of integers $0 = m_0 < m_1 < \dots$. Consider a diagram $B' = (V', E')$, where $V'_n = V_{m_n}$ and E'_n consists of all finite paths of B between levels $V_{m_{n-1}}$ and V_{m_n} . Then the diagram B' is called a *telescope* of B .

Observe that the telescoping procedure does not change the space of infinite paths and the full group of the diagram, see [HPS, Definition 2.2] for more details.

Using the minimality of the system (X, G) and Proposition 2.11, one can easily establish the following result.

Proposition 2.13. *Let G be the full group of a Bratteli diagram B . If G is a simple (non-trivial) group, then we can telescope a diagram B so that each pair of vertices from consecutive levels of the new diagram is always connected by an even (at least two) number of edges.*

Definition 2.14. Diagrams with the property as in the proposition above will be called *even diagrams*.

2.2 Inductive limits of symmetric groups

Fix a Bratteli diagram $B = (V, E)$ and consider the associated full group G . Denote by $S(M)$ the group of all permutations of a finite set M .

Denote by $\{F_n\}_{n \geq 1}$ the sequence of incidence matrices of B . For every level n and a vertex $v \in V_n$, consider the set $M_n^{(v)}$ of all finite paths of B connecting the root vertex v_0 and the vertex v . Clearly, the group $G_n^{(v)}$ can be identified with $S(M_n^{(v)})$. Then the structure of the diagram B gives a “natural” decomposition

$$M_n^{(v)} = \bigsqcup_{w \in V_{n-1}} \bigsqcup_{i=1}^{f_{v,w}^{(n)}} M_{n-1}^{(w)}, \quad (2)$$

where each set $M_{n-1}^{(w)}$ is taken with the multiplicity $f_{v,w}^{(n)}$. This decomposition, in its turn, gives a sequence of injective homomorphisms $\rho_n : G_{n-1} \rightarrow G_n$, $n \geq 1$. Thus, the group G can be seen as the inductive limit of the system $(G_{n-1} \xrightarrow{\rho_n} G_n)_{n \geq 1}$. Conversely, using arguments similar to Theorem 5.1 [LN] one can show that any inductive limit of symmetric groups with the embedding as above is isomorphic to the full group of some Bratteli diagram.

To avoid trivialities, we will always assume that Bratteli diagrams and the associated groups are not trivial and the path spaces are Cantor sets. Furthermore, we will assume that the system (X, G) is minimal.

2.3 Characters and factor representations

In this subsection we recall basic definitions from the representation theory and the theory of operator algebras. For details we refer the reader to the books [BR], [KR1], [KR2], and [Ta].

Let H be a separable Hilbert space with the inner product (\cdot, \cdot) . Denote by $L(H)$ the algebra of bounded linear operators on H . Each pair of nonzero vectors $\xi, \eta \in H$ determine semi-norms $\|A\|_{\xi, \eta} = |(A\xi, \eta)|$ and $\|A\|_{\xi} = \|A\xi\|$ on $L(H)$. In this paper, by an algebra of operators we always mean a $*$ -algebra containing the identity operator I .

Definition 2.15. The weak (strong) operator topology on $L(H)$ is the least topology making all the semi-norms $\|\cdot\|_{\xi, \eta}$ ($\|\cdot\|_{\xi}$) continuous.

For a subset of operators $\mathcal{S} \subset L(H)$ the *commutant* of \mathcal{S} is denoted by

$$\mathcal{S}' = \{A \in L(H) : AB = BA \text{ for each } B \in \mathcal{S}\}.$$

The commutant of a subset of operators is always an algebra closed in the weak and strong operator topologies. The commutant of \mathcal{S}' is denoted by \mathcal{S}'' . The proof of the following von Neumann bicommutant theorem can be found, for example, in [BR, Theorem 2.4.11].

Theorem 2.16. Let $\mathcal{A} \subset L(H)$ be a $*$ -subalgebra. Then the following conditions are equivalent:

- (1) \mathcal{A} is closed in the weak operator topology;
- (2) \mathcal{A} is closed in the strong operator topology;
- (3) $\mathcal{A} = \mathcal{A}''$.

Definition 2.17. A $*$ -subalgebra $\mathcal{A} \subset L(H)$ is called a *von Neumann (or W^* -) algebra* if it meets the conditions of Theorem 2.16.

Definition 2.18. A *unitary representation* of a group G is a homomorphism $G \rightarrow U(H)$, where $U(H)$ is the group of unitary operators on a Hilbert space H . If G is a topological group, the representation is called *continuous* if it is continuous with respect to the weak operator topology on $L(H)$.

For a unitary representation π of a group G denote by \mathcal{M}_{π} the W^* -algebra generated by the operators $\pi(G)$. Two unitary representations π_1 and π_2 of the same group G are called *quasi-equivalent* if there is an isomorphism of von Neumann algebras $\omega : \mathcal{M}_{\pi_1} \rightarrow \mathcal{M}_{\pi_2}$ such that $\omega(\pi_1(g)) = \pi_2(g)$ for each $g \in G$.

Definition 2.19. A representation π of a group G is called a *factor representation* if the algebra \mathcal{M}_{π} is a factor, that is $\mathcal{M}_{\pi} \cap \mathcal{M}'_{\pi} = \mathbb{C}I$.

Definition 2.20. A *character* of a group G is a function $\chi : G \rightarrow \mathbb{C}$ satisfying the following properties:

- (1) $\chi(g_1 g_2) = \chi(g_2 g_1)$ for any $g_1, g_2 \in G$;
- (2) the matrix $\{\chi(g_i g_j^{-1})\}_{i, j=1}^n$ is nonnegative-definite for any integer $n \geq 1$ and elements $g_1, \dots, g_n \in G$;
- (3) $\chi(e) = 1$, where e is the identity of G .

A character χ is called *indecomposable* if it cannot be represented in the form $\chi = \alpha\chi_1 + (1 - \alpha)\chi_2$, where $0 < \alpha < 1$ and χ_1, χ_2 are distinct characters.

Fix a character χ of the group G . Consider the space $L_0(G)$ of functions on G with finite supports. Define the bilinear form on $L_0(G)$ by

$$B_\chi(f_1, f_2) = \sum_{g_1, g_2 \in G} f_1(g_1) \overline{f_2(g_2)} \chi(g_1 g_2^{-1}).$$

It follows from Definition 2.20 that B_χ is a nonnegative-definite Hermitian form. Let \tilde{H}_χ be the completion of $L_0(G)$ with respect to B_χ . Set

$$H_\chi = \tilde{H}_\chi / \text{Ker}(B_\chi).$$

Then $H = H_\chi$ is a Hilbert space with the inner product (\cdot, \cdot) induced by B_χ . Define an action π of G on $L_0(G)$ by the formula

$$(\pi(g)f)(h) = f(g^{-1}h).$$

Observe that the form B_χ is invariant under $\pi(G)$. Therefore, π defines a unitary representation of G on H , which we denote by the same symbol π . Consider the function $\delta \in L_0(G)$ such that $\delta(e) = 1$ and $\delta(g) = 0$ for all $g \neq e$. Denote by $\xi \in H$ the image of δ in H . It follows that

$$(\pi(g)\xi, \xi) = B_\chi(\pi(g)\delta, \delta) = \chi(g) \text{ for any } g \in G.$$

One can show that the vector ξ is cyclic and separating for the algebra \mathcal{M}_π . Recall that ξ is *separating* if $A\xi = 0$, $A \in \mathcal{M}_\pi$, implies $A = 0$. The procedure described above is called the Gelfand-Naimark-Siegel (abbreviated “GNS”) construction. Using the disintegration theorem [Ta, Theorem 8.21], one can also establish the following result.

Proposition 2.21. *The representation π constructed above by χ is a factor representation if and only if the character χ is indecomposable.*

If χ is an indecomposable character, then the constructed von Neumann algebra \mathcal{M}_π is a finite-type factor, i.e. it is of type I_n , $n < \infty$, or of type II_1 , see definitions in [Ta, Chapter 5].

Conversely, consider a finite-type factor representation π of a group G on a Hilbert space. Then there is a unique positive linear functional tr on the algebra \mathcal{M}_π with the properties $tr(I) = 1$, $tr(AB) = tr(BA)$ for all $A, B \in \mathcal{M}_\pi$, and $tr(A^*A) > 0$ if $A \neq 0$, see [KR2, Theorem 8.2.8]. Applying the GNS construction (similar to the one above) to the algebra \mathcal{M}_π (see [KR1, Theorem 4.5.2]), we can find another realization of \mathcal{M}_π on a Hilbert space H such that $tr(A) = (A\xi, \xi)$ for some cyclic and separating vector $\xi \in H$. Hence $\chi(g) = tr(\pi(g)) = (\pi(g)\xi, \xi)$ is an indecomposable character on the group G . We can summarize the discussion above in the following result.

Theorem 2.22. (1) *Indecomposable characters are in one-to-one correspondence with finite-type factor representations.*

(2) *Two factor representations \mathcal{M}_{π_1} and \mathcal{M}_{π_2} constructed by characters χ_1 and χ_2 are quasi-equivalent if and only if $\chi_1 \equiv \chi_2$.*

Throughout the paper, the symbol χ will be reserved for a indecomposable character of some full group G . We will also use the notation (π, H, ξ) to denote a representation π , Hilbert space H , and a cyclic and separating vector $\xi \in H$ obtained by the GNS-construction for χ .

The following simple result shows that the continuity of the representation is equivalent to the continuity of the corresponding character.

Proposition 2.23. *Let G be a topological group. If a character $\chi : G \rightarrow \mathbb{C}$ is continuous, then the corresponding factor representation $\pi : G \rightarrow \mathcal{M}_\pi$ is continuous.*

Proof. For any $h_1, h_2 \in G$, the function

$$f_{h_1, h_2}(g) := (\pi(g)\pi(h_1)\xi, \pi(h_2)\xi) = \chi(h_2^{-1}gh_1)$$

is continuous in the group topology on G . Since $\text{Lin}\{\pi(h)\xi : h \in G\}$ is dense in the Hilbert space H , we get that for any $x, y \in H$ the function $f_{x, y}(g) = (\pi(g)x, y)$ is continuous on G . This implies that $\pi : G \rightarrow \mathcal{M}_\pi$ is continuous. \square

3 Algebraic Properties of Full Groups

In this section, we establish several algebraical results on full groups needed in consecutive sections. Additional algebraic and dynamical properties of full groups can be found in [GW], [GPS2], [M], and [BM]. Recall that the symbol G stands for the full group of a Bratteli diagram and X for its path-space. We always assume that the system (X, G) is minimal.

Definition 3.1. (1) For each element $g \in G$, set $\text{supp}(g) = \{x \in X : g(x) \neq x\}$ and $\text{Fix}(g) = \{x \in X : g(x) = x\}$, termed the *support* of g and the *set of fixed points* of g , respectively. Note that $\text{supp}(g)$ and $\text{Fix}(g)$ are clopen sets.

(2) We will denote by $\mathcal{M}(G)$ and $\mathcal{E}(G)$ the set of all probability G -invariant and probability ergodic G -invariant measures, respectively.

The following lemma is a crucial technical result that shows that if $\mu(A) < \mu(B)$ for all $\mu \in \mathcal{M}(G)$, then B contains a subset G -equivalent to the set A , for the proof see [GW, Lemma 2.5].

Lemma 3.2. *Let A and B be clopen subsets of X . If $\mu(A) < \mu(B)$ for all $\mu \in \mathcal{M}(G)$, then there is an involution $g \in G$ with $g(A) \subset B$ and $\text{supp}(g) \subset A \cup B$.*

The following lemma immediately follows from the minimality of the system (X, G) and the weak-* compactness of the set $\mathcal{M}(G)$.

Lemma 3.3. *For each clopen set $A \subset X$, $A \neq \emptyset$ the number $\delta(A) = \inf\{\mu(A) : \mu \in \mathcal{M}(G)\}$ is strictly positive.*

For any two elements $g, h \in G$, set

$$D(g, h) = \sup_{\mu \in \mathcal{E}(G)} \mu(\{x \in X : g(x) \neq h(x)\}).$$

Observe that the function D is invariant with respect to the group operations, i.e. $D(g, h) = D(qg, qh) = D(gq, hq) = D(g, h) = D(g^{-1}, h^{-1})$ for every $g, h, q \in G$, see, for example, the chapter on the uniform topology in [H]. Since for every clopen set A the value $\mu(A)$ is strictly positive for all $\mu \in \mathcal{M}(G)$, we get that $D(g, h) = 0$ implies $g = h$. Furthermore, using the same ideas as in [H], one can check that D satisfies the triangle inequality. This shows that G is a topological group with respect to the metric D . Following the traditions of ergodic theory, we give the following definition.

Definition 3.4. The topology generated by the metric D is called the *uniform topology*.

Definition 3.5. (1) We say that elements g and h are ε -conjugate if there is $q \in G$ with $D(qgq^{-1}, h) < \varepsilon$.

(2) We say that an element g *consists of even cycles* if the g -period of every point $x \in X$ is equal to one or to an even number. The word “even” refers to the length of the cycle and not to the parity of a permutation.

Given $n \geq 1$ and a clopen set A , define a subgroup $G_n(A)$ as the set of all elements $g \in G_n$ such that $\text{supp}(g) \subset A$. Notice that $G_n(A) \subset G_{n+1}(A)$. Set $G(A) = \bigcup_{n \geq 1} G_n(A)$. We will refer to groups $G(A)$ as to *local subgroups*. The following technical lemmas show connections between the algebraic structure of the full group and its action on the Bratteli diagram.

Lemma 3.6. *Let G be the full group of a Bratteli diagram B . Let A be a clopen subset of the path-space X . There exists a sequence of involutions $\{h_n\}_{n \geq 1} \subset G(A)$ satisfying the following conditions:*

- (i) h_n commutes with every element of $G_n(A)$;
- (ii) for every involution $s \in G_n(A)$ the elements sh_n and h_n are $1/n$ -conjugate;
- (iii) if an element $s \in G_n(A)$ with $\text{supp}(s) = A$ consists of even cycles, then the elements sh_n and s are conjugate;
- (iv) the elements $h_{n_1}h_{n_2}h_{n_3}$ and $h_{n_1}h_{n_2}$, $n_1 < n_2 < n_3$, are $1/n_1$ -conjugate;
- (v) the elements h_{n_1} and h_{n_2} , $n_1 < n_2$, are $1/n_1$ -conjugate.

All elements implementing the conjugations above are taken from the group $G(A)$.

If the full group G is simple, then, additionally, all $1/n$ -conjugations in the statements (ii) — (v) are conjugations. For example, the property (ii) states, in this case, that elements sh_n and h_n are, in fact, conjugate.

Proof. To simplify the notation, we will assume that the set A coincides with the entire space X and $G(A) = G$.

For $m > n$, $v \in V_n$, and $w \in V_m$ denote by $C_{v,w}$ the set of finite paths between v and w . By minimality, we can telescope the diagram B (denote the new diagram by the same symbol B) so that for each level n and vertices $v \in V_n$ and $w \in V_{n+1}$ we have that $\text{card}(C_{v,w}) > n$.

Fix an integer $n \geq 1$. For each pair of vertices $v \in V_n$ and $w \in V_{n+1}$, find two disjoint subsets $C_{v,w}^{(1)}, C_{v,w}^{(2)} \subset C_{v,w}$ of equal cardinality such that $\text{card}(C_{v,w} \setminus (C_{v,w}^{(1)} \cup C_{v,w}^{(2)})) \leq 1$. Let $h_{v,w} : C_{v,w} \rightarrow C_{v,w}$ be an involution ($h_{v,w}^2 = \text{id}$), which maps $C_{v,w}^{(1)}$ onto $C_{v,w}^{(2)}$. Denote the elements $h_n \in G_{n+1}$ as follows

$$h_n(e_1, \dots, e_n, e_{n+1}, e_{n+2}, \dots) = (e_1, \dots, e_n, h_{v,w}(e_{n+1}), e_{n+2}, \dots), \quad (3)$$

where (e_1, e_2, \dots) is any infinite path and $v = s(e_{n+1}), w = r(e_{n+1})$. We will show that the sequences $\{h_n\}_{n \geq 1}$ satisfies all the properties of the lemma. The construction of h_n implies that the element h_n commutes with all elements of G_n . This proves the property (i).

Recall that $G_n = \prod_{v \in V_n} G_n^{(v)}$ and $G_n^{(v)}$ is isomorphic to the group of permutations $S(M_n^{(v)})$ (see Subsection 2.2). Observe that two elements $g_1, g_2 \in G_n$ are conjugate if and only if for each $v \in V_n$ the corresponding permutations from $S(M_n^{(v)})$ are conjugate. In particular, if g_1 and g_2 are involutions, they are conjugate if and only if for each $v \in V_n$ the elements g_1 and g_2 have the same number of fixed paths in $M_n^{(v)}$.

Fix an involution $s \in G_n$. Then the element sh_n fixes a path (e_1, \dots, e_{n+1}) if and only if the permutation s fixes the path (e_1, \dots, e_n) and the involution $h_{v,w}$ fixes the edge e_{n+1} , where $v = s(e_n), w = r(e_{n+1})$. It follows from the definition of h_n that for each $w \in V_{n+1}$ the number of fixed paths in $M_{n+1}^{(w)}$ (for both h_n and sh_n) is at most $\text{card}(M_{n+1}^{(w)})/n$. Therefore, we can find an involution $h'_n \in G_{n+1}$ such that for each $w \in V_{n+1}$

- (1) h'_n differs from h_n on $M_{n+1}^{(w)}$ on at most $\text{card}(M_{n+1}^{(w)})/n$ paths;
- (2) h'_n has the same number of fixed paths in $M_{n+1}^{(w)}$ as sh_n .

Then h'_n is conjugate to sh_n . Fix a G -invariant measure μ on X . For a vertex $w \in W_{n+1}$, denote by X_w the set of all paths coming through the vertex $w \in W_{n+1}$. Set $\alpha_w = \mu(X_w)$. Observe that $\sum_{w \in W_{n+1}} \alpha_w = 1$ and $\mu([x_w]) = \alpha_w / \text{card}(M_{n+1}^{(w)})$ for any finite path from the root vertex v_0 to the vertex w . Here $[x_w]$ is the cylinder set of infinite paths coinciding with x_w .

Therefore,

$$\begin{aligned}
\mu(\{x \in X : h_n(x) \neq h'_n(x)\}) &= \sum_{w \in W_{n+1}} \mu(\{x \in X_w : h_n(x) \neq h'_n(x)\}) \\
&\leq \sum_{w \in W_{n+1}} \frac{\alpha_w}{\text{card}(M_{n+1}^{(w)})} \cdot \frac{\text{card}(M_{n+1}^{(w)})}{n} \\
&= 1/n.
\end{aligned}$$

This proves (ii).

Now let $s \in G_n$ consist of even cycles and $\text{supp}(s) = X$. Let $x = (x_1, \dots, x_{n+1})$ be a finite path and $x' = (x_1, \dots, x_n)$. Then, by definition of h_n , we have that

$$(sh_n)^l(x) = \begin{cases} (s^l(x'), h_{v,w}(x_{n+1})), & \text{if } l \text{ is odd,} \\ (s^l(x'), x_{n+1}), & \text{if } l \text{ is even,} \end{cases}$$

where $v = s(x_{n+1})$, $w = r(x_{n+1})$. Observe that the length of the orbit of x under s is even. It follows that the orbits of x under the actions of s and sh_n have the same lengths. Therefore, for any vertex $w \in V_{n+1}$ the elements s and sh_n induce conjugate permutations on $M_{n+1}^{(w)}$, which proves the property (iii).

The properties (iv) and (v) can be established similarly to (ii). We leave the details to the reader. \square

Lemma 3.7. *Let $s \in G$ and $A = \text{supp}(s)$. Then for any $r \in \mathbb{N}$ and any $\varepsilon > 0$ there exist elements $s_1, \dots, s_{2r} \in G$ satisfying the following conditions:*

- (1) *each element s_i is conjugate to s ;*
- (2) *$\text{supp}(s_i) = A$ and $\mu(A \setminus \text{supp}(s_i s_j^{-1})) < \varepsilon$ for any $1 \leq i \neq j \leq 2r$ and $\mu \in \mathcal{M}(G)$;*
- (3) *the element $s_i s_j^{-1}$, for any $i \neq j$, consists of even cycles (Definition 3.5).*

Proof. Denote by A_p the set of all points $x \in X$ such that $s^p(x) = x$ and $s^l(x) \neq x$ for $1 \leq l \leq p-1$. Then $X = \bigsqcup A_p$ is a clopen partition. Clearly, it is enough to prove the lemma for each restriction $s|_{A_p}$ and ensure that the elements $s_i^{(p)}$ are taken from $G(A_p)$. Then $s_i = \prod_p s_i^{(p)}$, $i = 1, \dots, 2r$, will be the desired elements. So, without loss of generality, we will assume that there exists an integer $p \geq 2$ such that $X = A_p$.

We will use the notation $(k_0, k_1, \dots, k_{l-1})$ to denote the cyclic permutation of symbols $\{k_0, \dots, k_{l-1}\}$. For example, (k, l) will stand for the transposition of k and l . Set also $M_n = \{0, 1, \dots, n-1\}$. The symmetric group on M_n will be denoted by $S(M_n)$. The proof of the lemma will be based on the following combinatorial observation.

Claim 1. *Let $p \geq 2$ be an integer. Set $m = 2p$ if p is even and $m = 2p-2$ if p is odd. There exists permutations $h_0, h_1 \in S(M_m)$ such that (i) each h_i is*

a cycle of length p ; (ii) each independent cycle of $h_0 h_1^{-1}$ is of even length; and (iii) $\text{card}(\text{supp}(h_0 h_1^{-1})) = m$.

Proof of the claim. If p is even, then set

$$h_0 = (0, 1, \dots, p-1) \text{ and } h_1 = (p, p+1, \dots, 2p-1).$$

The claim easily follows from the definition of h_0 and h_1 .

If p is odd, then define h_0 and h_1 as

$$h_0 = (0, 1, \dots, p-1) = (0, 1) \cdots (p-2, p-1)$$

and

$$h_1 = (2p-3, 2p-4, \dots, p-2) = (2p-3, 2p-4) \cdots (p-1, p-2).$$

Therefore,

$$\begin{aligned} h_0 h_1^{-1} &= (0, 1)(1, 2) \cdots (p-3, p-2)(p-2, p-1) \\ &\times (p-2, p-1)(p-1, p) \cdots (2p-2, 2p-3) \\ &= (0, 1, \dots, p-2) \times (p-1, p, \dots, 2p-3). \end{aligned}$$

Thus, $h_0 h_1^{-1}$ is a product of two cycles of length $p-1$.

Now the rest of the properties immediately follows from this construction.

This proves the claim.

Recall that V_n stands for the set of vertices of the n -th level. For each pair of vertices $v \in V_n$ and $w \in V_{n+1}$ denote by $E_{v,w}^{(n+1)}$ the set of all edges between v and w . Denote by $G_{n,n+1}$ the subgroup of all elements $g \in G$ such that g changes only the $n+1$ -st edge of x , $x \in X$, i.e. the edge between n -th and $n+1$ -st levels. Clearly, $G_{n,n+1} \subset G_{n+1}$.

Telescope the diagram B so that $s \in G_1$ and for any $n \geq 1$, any pair of vertices $v \in V_n, w \in V_{n+1}$ we have that

$$\left| E_{(v,w)}^{(n+1)} \right| > \frac{2p}{\varepsilon}. \quad (4)$$

Claim 2. Each group $G_{n,n+1}$, $n \geq 1$, contains two elements g_0, g_1 that satisfy the following conditions

- (i) the length of every nontrivial cycle of g_0 and g_1 is p ;
- (ii) all nontrivial cycles of $g_0 g_1^{-1}$ have even lengths;
- (iii) $\mu(\text{supp}(g_0 g_1^{-1})) > 1 - \varepsilon$ for each measure $\mu \in \mathcal{M}(G)$.

Proof of the claim. (a) For every pair of vertices $v \in V_n$ and $w \in V_{n+1}$, consider the set $E_{v,w}^{(n+1)}$. Set $m = 2p$ if p is even and $m = 2p-2$ if p is odd. Find non-negative integers t and q such that $\text{card}(E_{v,w}^{(n+1)}) = mt + q$, where

$q \in \{0, 1, \dots, m-1\}$. Note that the numbers t and q depend on the vertices v and w . The arguments below shall be independently applied to each pair of vertices. Leave q edges of the set $E_{v,w}^{(n+1)}$ out and partition the rest of the edges into t disjoint families F_0, \dots, F_{t-1} with $\text{card}(F_i) = m$. For each i , let $h_0^{(i)}$ and $h_1^{(i)}$ be permutations as in Claim 1 above. Note that the definition of $h_0^{(i)}$ and $h_1^{(i)}$ depends on the parity of the number p (as well as on the vertices v and w). Set $h_{v,w}^{(j)} = h_j^{(0)} \dots h_j^{(t-1)}$, $j = 0, 1$.

Thus, we have constructed permutations $h_{v,w}^{(0)}$ and $h_{v,w}^{(1)}$ of the set $E_{v,w}^{(n+1)}$ such that each permutation $h_{v,w}^{(j)}$, $j = 0, 1$, consists of disjoint cycles of length p and

$$\text{card} \left(h_{v,w}^{(0)} (h_{v,w}^{(1)})^{-1} \right) = t \cdot m. \quad (5)$$

Define a homeomorphism $g_i : X \rightarrow X$ by

$$g_i(x_1, \dots, x_{n-1}, x_n, \dots) = (x_1, \dots, x_{n-1}, h_{v,w}^{(i)}(x_n), \dots) \text{ if } x_n \in E_{v,w}^{(n+1)}. \quad (6)$$

Clearly, g_0 and g_1 are cycles of length p belonging to $G_{n,n+1}$. It follows from Equations (4)-(6) that for each pair of vertices $v \in V_n, w \in V_{n+1}$ the homeomorphism $g_0 g_1^{-1}$ acts on all, but less than $\varepsilon \left| E_{v,w}^{(n+1)} \right|$ edges from $E_{v,w}^{(n+1)}$. Therefore,

$$\mu(\text{supp}(g_0 g_1^{-1})) > 1 - \varepsilon \text{ for every invariant measure } \mu.$$

This establishes the condition (iii) of the claim. The rest of the conditions follows easily from the definition of g_0 and g_1 . *This proves the claim.*

For each $n \geq 1$, take homeomorphisms $g_{n+1}^{(0)}, g_{n+1}^{(1)} \in G_{n,n+1}$ satisfying all conditions of Claim 2 above. For each tuple $a = (a_1, a_2, \dots, a_r)$, $a_i \in \{0, 1\}$, define a homeomorphism $s_a^{(i)} \in G_{1+r}$ by

$$s_a(x) = s g_2^{(a_1)} \cdot g_3^{(a_2)} \dots g_{r+1}^{(a_r)}(x),$$

where $x \in X$. Recall that $s \in G_1$. Since all homeomorphisms $\{g_{i+1}^{(a_i)}\}$ act on different levels (starting with $n \geq 2$) and consist of cycles of length p , we conclude that s_a is conjugate to s for every $a \in \{0, 1\}^r$. Moreover, for distinct $a, b \in \{0, 1\}^r$ we have that

$$s_a s_b^{-1}(x) = g_2^{(a_1)} (g_2^{(b_1)})^{-1} \dots g_{r+1}^{(a_r)} (g_{r+1}^{(b_r)})^{-1}(x), \quad x \in X.$$

Choose $i = 1, \dots, r$ such that $a_i \neq b_i$. Then $\text{supp}(s_a s_b^{-1}) \supset \text{supp}(g_{i+1}^{(a_i)} (g_{i+1}^{(b_i)})^{-1})$ and $\mu(\text{supp}(s_a s_b^{-1})) > 1 - \varepsilon$ for every $\mu \in \mathcal{M}(G)$.

To complete the proof of the lemma, we observe only that the homeomorphism $s_a s_b^{-1}$ consists of cycles of length p if p is even and of length $p-1$ if p is odd. \square

4 Orthogonal Projections

In the section we define a family of orthogonal projections, which, in some sense, behave as operators of multiplication by characteristic functions. Here we also establish the first main result of the paper (Theorem 1.1). Fix an indecomposable character χ of the group G and the corresponding factor representation $\pi : G \rightarrow \mathcal{M}_\pi$, see Section 2.3. The symbol H stands for the Hilbert space on which the algebra \mathcal{M}_π acts and ξ stands for the cyclic and separating vector for the algebra \mathcal{M}_π . Recall that $\text{tr}(Q) = (Q\xi, \xi)$ for all $Q \in \mathcal{M}_\pi$ and $\chi(g) = \text{tr}(\pi(g))$ for all $g \in G$.

Definition 4.1. Fix a clopen set $A \subset X$. (1) Denote by P^A the orthogonal projector onto the space

$$H^A = \{h \in H : \pi(g)h = h \text{ for all } g \in G(A)\}.$$

Remark 4.2. (1) Since the character χ is indecomposable, the existence of a non-zero invariant vector for the group $\pi(G)$ implies that the representation π is trivial, i.e. $\pi(g) = I$ is the identity operator for every $g \in G$. This implies that $\chi(g) = 1$ for all g . If $\chi \neq 1$, then there are no invariant vectors, which implies that $P^X = 0$.

(2) We will use the convention that $P^\emptyset = I$ is the identity operator.

(3) Notice that it is not clear a priori that $P^A \neq 0$ for at least one clopen set $A \neq \emptyset$.

(4) If $g \in G(A)$, then $\pi(g)P^A = P^A$.

(5) It will follow from Proposition 4.3 that the projection P^A belongs to \mathcal{M}_π the von Neumann algebra generated by $\pi(G)$.

We recall that the projector P_1 *dominates* P_2 , in symbols $P_1 \geq P_2$, if $P_2(H) \subset P_1(H)$. This, in particular, implies that $\text{tr}(P_1) \geq \text{tr}(P_2)$. Recall that when we talk about limits in \mathcal{M}_π , we always mean *weak limits*.

Proposition 4.3. *Let B be a Bratteli diagram and G the associated full group. Assume that either the group G is simple or the representation $\pi : G \rightarrow \mathcal{M}_\pi$ is continuous. Let A be a clopen subset of the path-space X and $\{h_n\}_{n \geq 1}$ be a sequence of involutions satisfying the properties of Lemma 3.6 for the set A . Then*

$$P^A = \lim_{n \rightarrow \infty} \pi(h_n).$$

Proof. We will simultaneously consider the cases of simple groups (even diagrams, see Proposition 2.11) and continuous representations. Note that if the representation π is continuous, then the corresponding character χ is also continuous.

Consider a sequence $\{h_n\}_{n \geq 1}$ satisfying all the statements of Lemma 3.6 and the corresponding sequence of unitary operators $\{\pi(h_n)\}_{n \geq 1}$. Passing to a subsequence (we will drop the extra subindex), we can assume that $\{\pi(h_n)\}$ weakly converges to some operator $Q \in \mathcal{M}_\pi$.

Claim 1. Q is an orthogonal projector. Since all elements $\{h_n\}$ are involutions, we obtain that $\pi(h_n)^* = \pi(h_n^{-1}) = \pi(h_n)$. Hence $Q = Q^*$ is a self-adjoint operator.

Fix a triple $n_1 < n_2 < n_3$. Lemma 3.6(iv) implies that $h_{n_1}h_{n_2}h_{n_3}$ and $h_{n_1}h_{n_2}$ are either conjugate (for simple groups) or $1/n_1$ -conjugate (for continuous representations), $n_1 < n_2 < n_3$. In either case, we have

$$|tr(\pi(h_{n_1})Q^2) - tr(\pi(h_{n_1})Q)| = \lim_{n_2 \rightarrow \infty} \lim_{n_3 \rightarrow \infty} |\chi(h_{n_1}h_{n_2}h_{n_3}) - \chi(h_{n_1}h_{n_2})|.$$

Therefore

$$|tr(\pi(h_{n_1})Q^2) - tr(\pi(h_{n_1})Q)| \rightarrow 0 \text{ as } n_1 \rightarrow \infty.$$

This implies that $tr(Q^3) = tr(Q^2)$, i.e.

$$(Q^3\xi, \xi) = (Q^2\xi, \xi) = (Q\xi, Q\xi) = \|Q\xi\|^2. \quad (7)$$

Using the Cauchy-Schwarz inequality, we get that

$$\|Q\xi\|^2 = (Q^3\xi, \xi) = (Q^2\xi, Q\xi) \leq \|Q\xi\| \cdot \|Q^2\xi\| \leq \|Q\xi\| \cdot \|Q\xi\| = \|Q\xi\|^2,$$

i.e., $(Q^2\xi, Q\xi) = \|Q\xi\| \cdot \|Q^2\xi\|$. Therefore, the vectors $Q\xi$ and $Q^2\xi$ are linearly dependent. Since the vector ξ is separating for the algebra \mathcal{M}_π , we get that $Q^2 = cQ$ for some $c \in \mathbb{C}$. Using (7), we see that $\|Q\xi\|^2 = (Q^2\xi, Q\xi) = c(Q\xi, Q\xi) = c\|Q\xi\|^2$. This implies that $c = 1$ and Q is a projector.

Claim 2. $\pi(s)Q = Q$ for every $s \in G(A)$. Since every element of $G(A)$ can be represented as a product of involutions, it suffices to establish the claim for involutions only. Fix an involution $s \in G(A)$. For all n large enough, the element sh_n and h_n are either conjugate (for simple groups) or $1/n$ -conjugate (for continuous representations), see Lemma 3.6(ii). It follows that $tr(\pi(s)Q) = tr(Q)$, i.e.

$$(\pi(s)Q\xi, \xi) = tr(\pi(s)Q) = tr(Q) = (Q\xi, \xi) = (Q^2\xi, \xi) = (Q\xi, Q\xi) = \|Q\xi\|^2.$$

Using the Cauchy-Schwarz inequality, we get that

$$\|Q\xi\|^2 = (\pi(s)Q\xi, \xi) = (\pi(s)Q\xi, Q\xi) \leq \|\pi(s)Q\xi\| \cdot \|Q\xi\| \leq \|Q\xi\|^2.$$

Since we have the equality here, the vectors $\pi(s)Q\xi$ and $Q\xi$ are linearly dependent. It is easy to check that, in fact, $\pi(s)Q\xi = Q\xi$. Since the vector ξ is separating, we conclude that $\pi(s)Q = Q$.

Claim 3. $Q = P^A$. Since $h_n \in G(A)$, we have that $\pi(h_n)P^A = P^A$. Therefore, $QP^A = P^A$, i.e. $P^A \leq Q$. On the other hand, Claim 2 above implies that every vector from the subspace QH is a fixed vector for $\pi(s)$, for any $s \in G(A)$. This implies that $QH \subset P^AH$ (see the definition of P^A), i.e. $Q \leq P^A$. It follows that $Q = P^A$. Thus, the sequence $\pi(h_n)$ has a unique weak limit point, which coincides with P^A . Therefore,

$$P^A = \lim_{n \rightarrow \infty} \pi(h_n).$$

□

The next proposition shows that the projectors $\{P^A\}$ form an Abelian semi-group.

Proposition 4.4. *Let B be a Bratteli diagram and G the associated full group. Assume that either G is simple or the representation of G is continuous. Let A_1 and A_2 be clopen subsets of X .*

- (1) *If $A_1 \subset A_2$, then $P^{A_1} \geq P^{A_2}$ and $\text{tr}(P^{A_1}) \geq \text{tr}(P^{A_2})$.*
- (2) *For any $g \in G$, we have that $\pi(g)P^{A_1}\pi(g^{-1}) = P^{g(A_1)}$ and $\text{tr}(P^{A_1}) = \text{tr}(P^{g(A_1)})$.*
- (3) *$P^{A_1}P^{A_2} = P^{A_1 \cup A_2}$.*

Proof. (1) Note that if $A_1 \subset A_2$, then $G(A_1) \subset G(A_2)$. It follows that $H^{A_2} \subset H^{A_1}$ and $P^{A_2} \leq P^{A_1}$.

(2) Fix an element $g \in G$. Clearly, $\pi(g)P^{A_1}\pi(g^{-1})$ is a projector onto $\pi(g)H^{A_1}$. Since $\pi(g)H^{A_1}$ consists of vectors invariant under $\pi(g)\pi(q)\pi(g^{-1})$, $q \in G(A_1)$, and $gG(A_1)g^{-1} = G(g(A_1))$, we conclude that $\pi(g)P^{A_1}\pi(g^{-1}) = P^{g(A_1)}$.

(3-i) First assume that clopen sets A_1 and A_2 are disjoint. Find the sequences $\{h_n^{(i)}\} \subset G(A_i)$ of involutions satisfying all the properties of Lemma 3.6 for the set A_i , $i = 1, 2$. Observe that the sequence (or some subsequence) $h_n = h_n^{(1)}h_n^{(2)}$, $n \geq 1$, also meet all the properties of Lemma 3.6 for the set $A_1 \cup A_2$. Thus, passing to subsequences and using Proposition 4.3, we conclude that

$$\begin{aligned} P^{A_1}P^{A_2} &= \lim_{n \rightarrow \infty} P^{A_1}\pi(h_n^{(2)}) = \lim_{n \rightarrow \infty} P^{A_1}\pi(h_n^{(1)}h_n^{(2)}) = P^{A_1}P^{A_1 \cup A_2} \\ &= \lim_{n \rightarrow \infty} \pi(h_n^{(1)})P^{A_1 \cup A_2} = P^{A_1 \cup A_2}. \end{aligned}$$

(3-ii) Now consider arbitrary clopen sets A_1 and A_2 . Set $C = A_1 \cap A_2$. It follows from (3-i) that

$$P^{A_1 \setminus C}P^C P^{A_2 \setminus C} = P^{A_1}P^{A_2 \setminus C} = P^{A_1 \cup A_2}.$$

This completes the proof. \square

Let A and B be clopen sets. Set $d(A, B) = \sup\{\mu(A \triangle B) : \mu \in \mathcal{M}(G)\}$. Observe that $d(A, B) \leq d(A, C) + d(C, B)$ for any clopen set C . The following lemma shows that the trace of the projector P^A depends “continuously” on a clopen set A . We would like to emphasize that all sets in this lemma are assumed to be non-empty.

Lemma 4.5. *For every clopen set A and $\varepsilon > 0$, there exists $\delta > 0$ such that if a clopen set B satisfies $d(A, B) < \delta$, then $|\text{tr}(P^A) - \text{tr}(P^B)| < \varepsilon$.*

Proof. Choose an integer m such that $2/m < \varepsilon$. Setting $C = A \cap B$, it suffices to show that there exists $\delta > 0$ such that if $d(Z, C) < \delta$, for $Z = A$ and $Z = B$, then $|\text{tr}(P^Z) - \text{tr}(P^C)| < 1/m$. We will only show how to choose δ for $Z = A$. The proof for the set B is analogous.

Set

$$\delta = \frac{\inf\{\mu(A) : \mu \in \mathcal{M}(G)\}}{2m}.$$

It follows that if $C \subset A$ with $d(A, C) < \delta$, then $\mu(C) \geq \mu(A)/2$ and

$$\mu(A \setminus C) < \delta \leq \frac{\mu(A)}{2m} \leq \frac{\mu(C)}{m}$$

for every measure $\mu \in \mathcal{M}(G)$. Therefore, using Lemma 3.2, we can find involutions $s_j \in G(A)$ with $s_i(A \setminus C) \subset C$, $j = 1, \dots, m$, and

$$s_j(A \setminus C) \cap s_l(A \setminus C) = \emptyset \text{ whenever } j \neq l.$$

This implies that $s_j(C) \cup s_l(C) = A$ for $j \neq l$. Thus, by Proposition 4.4, we obtain that ($j \neq l$)

$$\begin{aligned} & (P^{s_j(C)} - P^A) (P^{s_l(C)} - P^A) \\ &= P^{s_j(C)} P^{s_l(C)} - P^{s_j(C)} P^A - P^A P^{s_l(C)} + P^A P^A \\ &= P^A - P^A - P^A + P^A = 0. \end{aligned}$$

Therefore, $\{P^{s_j(C)} - P^A\}_{j=1}^m$ is a family of mutually orthogonal projectors. It follows that $\sum_{j=1}^m (P^{s_j(C)} - P^A)$ is a projector. Hence,

$$1 \geq \sum_{j=1}^m \text{tr} (P^{s_j(C)} - P^A) = \sum_{j=1}^m \text{tr} (\pi(s_j)(P^C - P^A)\pi(s_j^{-1})) = m \cdot \text{tr} (P^C - P^A).$$

It follows that $0 \leq \text{tr} (P^C - P^A) \leq 1/m$. \square

Now we are ready to establish the first main result of the paper.

Theorem 4.6. *Let B be a Bratteli diagram and G the associated full group. Assume that either the group G is simple or the representation $\pi : G \rightarrow M_\pi$ is continuous with respect to the uniform topology on G . Let χ be the character corresponding to the representation π . Then*

$$\chi(g) = \text{tr}(P^{\text{supp}(g)}) \text{ for every } g \in G.$$

Proof. (1) First of all, we will prove the result when g consists of even cycles (see Definition 3.5). Take a sequence $\{h_n\}_{n \geq 1}$ satisfying the properties of Lemma 3.6 for the set $A = \text{supp}(g)$. It follows from Lemma 3.6(iii) that the elements gh_n and g are conjugate, assuming that n is large enough. Then $\chi(gh_n) = \chi(g)$. Applying Proposition 4.3 and Remark 4.2(4), we get that

$$\chi(g) = \lim_{n \rightarrow \infty} \chi(gh_n) = \lim_{n \rightarrow \infty} (\pi(g)\pi(h_n)\xi, \xi) = (\pi(g)P^A\xi, \xi) = (P^A\xi, \xi) = \text{tr}(P^A).$$

(2) Now, consider the case when g is an arbitrary non-identity element of G . Set $A = \text{supp}(g)$. Fix an arbitrary integer $r \geq 1$ and arbitrary $\varepsilon > 0$. Using Lemma 4.5, choose $\delta > 0$ such that $|\text{tr}(P^A) - \text{tr}(P^B)| < \varepsilon/r^2$ for any clopen set $B \subset A$ with $d(A, B) < \delta$. By Lemma 3.7 there exist elements $s_1, \dots, s_r \in G$ such that

- (i) each element s_i is conjugate to g ;
- (ii) $A = \text{supp}(s_i)$ and $d(\text{supp}(s_i s_j^{-1}), A) < \delta$ for any $i \neq j$;
- (iii) the element $s_i s_j^{-1}$ consists of even cycles, when $i \neq j$.

Consider the family of vectors $\eta_i = (\pi(s_i) - P^A)\xi$. Put also $\eta = (\pi(g) - P^A)\xi$.

Since $s_i \in G(A)$, we get that $\pi(s_i)P^A = P^A$. Thus, for any $i \neq j$ we have that

$$\begin{aligned}
|(\eta_i, \eta_j)| &= |((\pi(s_i) - P^A)\xi, (\pi(s_j) - P^A)\xi)| \\
&= |(\pi(s_i)\xi, \pi(s_j)\xi) - (\pi(s_i)\xi, P^A\xi) - (P^A\xi, \pi(s_j)\xi) + (P^A\xi, P^A\xi)| \\
&= |\chi(s_i s_j^{-1}) - \text{tr}(P^A) - \text{tr}(P^A) + \text{tr}(P^A)| \\
&= |\text{tr}(P^{\text{supp}(s_i s_j^{-1})}) - \text{tr}(P^A)| \\
&\leq \varepsilon/r^2.
\end{aligned}$$

Using similar arguments, we obtain that $\|\eta_i\|^2 = 1 - \text{tr}(P^A) = \|\eta\|^2$ for every $i = 1, \dots, r$. Since s_i is conjugate to g , we see that $(\eta_i, \xi) = (\eta, \xi)$ for every i . It follows that

$$|(\eta, \xi)| = \frac{1}{r} \left| \left(\sum_{i=1}^r \eta_i, \xi \right) \right| \leq \frac{1}{r} \left\| \sum_{i=1}^r \eta_i \right\| = \frac{1}{r} \sqrt{\sum_{i,j=1}^r (\eta_i, \eta_j)} \leq \frac{\sqrt{r\|\eta\| + \varepsilon}}{r}.$$

Since r and ε are chosen arbitrarily, we conclude that $(\eta, \xi) = 0$. It follows that $\chi(g) = \text{tr}(P^A)$. \square

5 Formula for the Character

In this section we prove that each indecomposable character χ of the group G is defined as $\chi(g) = \mu_1(\text{Fix}(g))^{\alpha_1} \dots \mu_k(\text{Fix}(g))^{\alpha_k}$, where μ_1, \dots, μ_k are ergodic measures and $\alpha_1, \dots, \alpha_k \in \{0, 1, \dots, \infty\}$. *From now on, we assume that the set of ergodic measures $\mathcal{E}(G)$ is finite. Enumerate all ergodic measures of (X, G) as $\{\mu_1, \dots, \mu_k\}$.*

Lemma 5.1. *For any sequences $(x_1, \dots, x_k) \in [0, 1]^k$ and $(y_1, \dots, y_k) \in (0, 1]^k$, any $\varepsilon > 0$, and any $n \geq 1$, we can find clopen G_n -invariant sets A and B such that $x_i \leq \mu_i(A) < x_i + \varepsilon$ and $y_i - \varepsilon < \mu_i(B) \leq y_i$ for all $i = 1, \dots, k$.*

Proof. (1) Fix the subgroup G_n . Since all ergodic measures are mutually singular, we can find a disjoint family of G -invariant Borel sets B_1, \dots, B_k with $\mu_i(B_i) = 1$, for $i = 1, \dots, k$. For every $i = 1, \dots, k$, choose a closed set $C_i \subset B_i$ with

$$x_i \leq \mu_i([C_i]_n) < x_i + \varepsilon/3, \text{ where } [C_i]_n := \bigcup_{g \in G_n} g(C_i).$$

Notice that each set $[C_i]_n$ is closed and G_n -invariant.

Since $[C_i]_n \cap [C_j]_n = \emptyset$ for $i \neq j$, we can find a disjoint family of clopen sets $\{A_i\}_{i=1}^k$ such that $[C_i]_n \subset A_i$, $\mu_i(A_i) < \mu_i([C_i]_n) + \varepsilon/3$ and $\mu_i(\bigcup_{j \neq i} A_j) < \varepsilon/3$, $i = 1, \dots, k$. Setting

$$A = \bigcup_{i=1}^k \bigcap_{g \in G_n} g(A_i),$$

we get that A is a clopen G_n -invariant set that gives the desired approximation.

(2) Consider $(y_1, \dots, y_k) \in (0, 1]^k$. Define a sequence $x = (1 - y_1, \dots, 1 - y_k) \in [0, 1)^k$. Find a clopen G_n -invariant set A as in the part (1). Set $B = X \setminus A$. \square

Define a map F from the family of clopen sets into $[0, 1]^k$ by

$$F(A) = (\mu_1(A), \dots, \mu_k(A)),$$

where A is a clopen set. Set also

$$M = \{F(A) : A \text{ is a proper clopen set}\} \subset (0, 1)^k.$$

Equip \mathbb{R}^k with the sup-norm $\|c\| = \sup_{i=1, \dots, k} \|c_i\|$.

Lemma 5.2. *For every $c \in [0, 1]^k$ and $\varepsilon > 0$, there exists $\delta > 0$ such that if non-empty clopen sets A and B satisfy the conditions $\|F(A) - c\| < \delta/2$ and $\|F(B) - c\| < \delta/2$, then $|tr(P^A) - tr(P^B)| < \varepsilon$.*

Proof. Choose an integer m such that $2/m < \varepsilon$. Set

$$\delta = \frac{\min\{c_i : i = 1, \dots, k\}}{3m}.$$

Let A and B be clopen sets satisfying $\|F(A) - c\| < \delta/2$ and $\|F(B) - c\| < \delta/2$. Then $\mu_i(A) - \delta < \mu_i(B)$ for each $i = 1, \dots, k$. Using Lemmas 5.1 and 3.2, find a clopen subset $A' \subset A$ such that for each $i = 1, \dots, k$

$$\mu_i(A) - \delta < \mu_i(A') < \mu_i(B).$$

By Lemma 3.2 there exists an element $g \in G$ such that $g(A') \subset B$. Clearly, $d(g(A), B) < \delta$. Observe that

$$\delta < \frac{\min\{\mu_i(A), \mu_i(B)\}}{2m}$$

for each $i = 1, \dots, k$. Now using Proposition 4.4(2) and repeating the arguments of Lemma 4.5 we obtain that $|tr(P^A) - tr(P^B)| = |tr(P^{g(A)}) - tr(P^B)| < \varepsilon$. \square

Fix an arbitrary sequence $x = (x_1, \dots, x_k) \in (0, 1]^k$. Then there exists a sequence of clopen sets $\{A_n\}_{n \geq 1}$ such that $x = \lim_{n \rightarrow \infty} F(A_n)$. Lemma 5.2 applied to the sequence $\{F(X \setminus A_n)\}_{n \geq 1}$ implies that the sequence $\{tr(P^{X \setminus A_n})\}_{n \geq 1}$ is convergent and the limit does not depend on the choice of $\{A_n\}_{n \geq 1}$. Set

$$\varphi(x) = \lim_{n \rightarrow \infty} tr(P^{X \setminus A_n}).$$

Remark 5.3. Notice that a priori it is not clear that $\varphi(1, \dots, 1) = 1$. This will later follow from the multiplicativity property of the function φ provided the character χ is not regular, see Proposition 5.5.

Lemma 5.4. *Let μ be a G -invariant ergodic measure. Let B_n be a G_n -invariant clopen set such that $\mu(B_n) \rightarrow \mu(B) > 0$ as $n \rightarrow \infty$, where B is some clopen set. Then for any clopen set A we have that $\mu(A \cap B_n) \rightarrow \mu(A)\mu(B)$ as $n \rightarrow \infty$.*

Proof. Consider a sequence of probability measures $\{\mu_n\}$ given by

$$\mu_n(C) := \frac{\mu(C \cap B_n)}{\mu(B_n)}.$$

Since the set of all probability measures on X is weakly compact, we can find a subsequence $\{n_l\}$ such that the sequence $\{\mu_{n_l}\}$ weakly converges to some probability measure ν . It will follow from the proof below that ν is the only accumulation point of the sequence $\{\mu_n\}$. Thus, the measures μ_n converge to ν . To simplify the notation, we will drop the extra index l .

Observe that μ_n is G_n -invariant. Indeed, for $g \in G_n$ and a Borel set C , we have that

$$\mu_n(g(C)) = \frac{\mu(g(C) \cap B_n)}{\mu(B_n)} = \frac{\mu(g(C \cap B_n))}{\mu(B_n)} = \frac{\mu(C \cap B_n)}{\mu(B_n)} = \mu_n(C).$$

This implies that the limiting measure ν is G -invariant. Note also that $\nu(C) \leq \mu(C)/\mu(B)$ for any closed set (and thus for any set) C . Set $\delta = \mu(B)$. Hence $\mu = \delta\nu + (\mu - \delta\nu)$. Since $\mu - \delta\nu$ is a (non-negative) G -invariant measure, the ergodicity of μ implies that $\mu = \nu$.

Using the definition of ν , we obtain that

$$\mu_n(A) = \frac{\mu(B_n \cap A)}{\mu(B_n)} \rightarrow \mu(A).$$

Thus, $\mu(B_n \cap A) \rightarrow \mu(A)\mu(B)$. □

For two sequences $x = (x_i)$ and $y = (y_i)$ from $[0, 1]^k$ we will write $x \leq y$ if $x_i \leq y_i$ for every i . The following proposition shows that φ is a continuous, monotone, and multiplicative function. Notice that the multiplicativity of φ can be seen as a version of the multiplicativity of characters for $S(\infty)$.

Proposition 5.5. (1) *The function $\varphi : (0, 1]^k \rightarrow [0, 1]$ is continuous.*

(2) *The function φ is monotone, in the sense that, $a \leq b$, $a, b \in (0, 1]^k$, implies $\varphi(a) \leq \varphi(b)$.*

(3) *The function φ is multiplicative, i.e. for $a, b \in (0, 1]^k$ one has that*

$$\varphi(a \cdot b) = \varphi(a)\varphi(b), \text{ where } a \cdot b = (a_1b_1, \dots, a_kb_k).$$

Proof. (1) The continuity of φ immediately follows from the definition of φ and Lemma 5.2.

(2) First assume that $a, b \in M$ and $a < b$. Choose clopen sets A and B such that $a = F(A)$ and $b = F(B)$. By Lemma 3.2 there exists an element $g \in G$ such that $g(A) \subset B$. It follows that

$$\varphi(a) = \text{tr}(P^{X \setminus A}) \leq \text{tr}(P^{X \setminus B}) = \varphi(b).$$

The density of M in $(0, 1]^k$ and continuity of φ imply that $\varphi(a) \leq \varphi(b)$ for any $a, b \in (0, 1]^k$ with $a \leq b$.

(3) Let $a, b \in (0, 1]^k$. Assume first of all that $a \in M$. Let A be a clopen set such that $a = F(A)$. Using Lemma 5.1, we can find a sequence $\{B_n\}_{n \geq 1}$ of clopen sets with $\lim_{n \rightarrow \infty} F(B_n) = b$ and B_n being G_n -invariant.

Lemma 5.4 implies that, for every $i = 1, \dots, k$, $\lim_{n \rightarrow \infty} \mu_i(A \cap B_n) = a_i b_i$. By the continuity of φ , we have that

$$\varphi(a \cdot b) = \lim_{n \rightarrow \infty} \varphi(F(A \cap B_n)).$$

Consider a sequence of projections $\{P^{X \setminus B_n}\}_{n \geq 1}$. Passing to a subsequence (we keep the same notation), we can assume that $P^{X \setminus B_n}$ is weakly convergent to an operator P , see [KR1, Theorem 5,1,3]. The G_n -invariance of the set $X \setminus B_n$ implies that the projection $P^{X \setminus B_n}$ commutes with all operators of the form $\pi(g)$, $g \in G_n$, see Proposition 4.4. Therefore, the operator P belongs to the center of the algebra \mathcal{M}_π and, hence, is a scalar operator $P = cI$. Then by the continuity of φ , we get that

$$\varphi(b) = \lim_{n \rightarrow \infty} \varphi(F(B_n)) = \lim_{n \rightarrow \infty} \text{tr}(P^{X \setminus B_n}) = \text{tr}(P) = c.$$

Using Proposition 4.4, we obtain that

$$\begin{aligned} \varphi(F(A \cap B_n)) &= \text{tr}(P^{X \setminus (A \cap B_n)}) \\ &= \text{tr}(P^{X \setminus A} P^{X \setminus B_n}) \\ &\rightarrow \text{tr}(P^{X \setminus A}) \varphi(b) \text{ (as } n \rightarrow \infty) \\ &= \varphi(a) \varphi(b). \end{aligned}$$

Therefore, $\varphi(a \cdot b) = \varphi(a) \varphi(b)$ for all $a \in M$ and $b \in (0, 1]^k$. The continuity of φ implies that $\varphi(a \cdot b) = \varphi(a) \varphi(b)$ for all $a, b \in (0, 1]^k$. \square

Now we are ready to establish the main result of the paper. Recall that the character χ is regular if $\chi(g) = 0$ for all group elements g different from the identity.

Theorem 5.6. *Let G be the full group of a Bratteli diagram. Let $\chi : G \rightarrow \mathbb{C}$ be a non-zero indecomposable character of G . Assume that the action of G on the Bratteli diagram has only a finitely many ergodic measures, say μ_1, \dots, μ_k , and that either the group G is simple or the character χ is continuous with respect to the uniform topology. Then there exist $\alpha_1, \dots, \alpha_k \in \{0, 1, \dots, \infty\}$ such that*

$$\chi(g) = \mu_1(\text{Fix}(g))^{\alpha_1} \mu_2(\text{Fix}(g))^{\alpha_2} \cdots \mu_k(\text{Fix}(g))^{\alpha_k} \quad (8)$$

for any $g \in G$. If some $\alpha_i = \infty$, then the character χ is regular.

Proof. (1) For $i = 1, \dots, k$ and $x \in \mathbb{R}$, set $e_i(x) = (1, \dots, 1, x, 1, \dots, 1)$, where x stands on the i -th position. Define a function $\varphi_i : (0, 1] \rightarrow [0, 1]$ by $\varphi(x) = \varphi(e_i(x))$. Proposition 5.5 implies that

$$\varphi_i(x \cdot y) = \varphi(e_i(x \cdot y)) = \varphi(e_i(x) \cdot e_i(y)) = \varphi_i(x) \cdot \varphi_i(y) \text{ for all } x, y \in (0, 1].$$

If $\varphi_i(x) = 0$ for some $x \in (0, 1]$, then the multiplicativity of φ_i implies that $\varphi_i(y) = 0$ for every $y \in (0, 1]$. Since $\varphi((x_1, \dots, x_k)) = \varphi_1(x_1) \cdots \varphi_k(x_k)$, we get that $\varphi(x) = 0$ for every $x \in (0, 1]$ and $\chi(g) = 0$, for all $g \neq 1$, by Theorem 4.6. Thus, the character χ is regular.

If χ is not regular, then $\varphi_i(x) \neq 0$ for all $x \in (0, 1]$ and $i = 1, \dots, k$. Set $f_i(t) = -\log \varphi_i(e^{-t})$. Then f_i is continuous and additive on $\mathbb{R}_+ = \{t > 0 : t \in \mathbb{R}\}$. Thus, solving Cauchy's functional equation, we can find a real number $\alpha_i \geq 0$ such that $f_i(t) = \alpha_i t$ for all $t \in \mathbb{R}_+$. Therefore, $\varphi_i(x) = x^{\alpha_i}$ for every $x \in (0, 1]$. Thus,

$$tr(P^A) = \varphi(F(X \setminus A)) = \mu_1(X \setminus A)^{\alpha_1} \mu_2(X \setminus A)^{\alpha_2} \cdots \mu_k(X \setminus A)^{\alpha_k}$$

for any clopen set A . Now, the formula (8) follows from Theorem 4.6.

(2) It only remains to show that the parameters $\alpha_1, \dots, \alpha_k$ are integers. Fix $1 \leq i \leq k$ and $\epsilon > 0$. By Lemma 5.1, there exists a clopen set A such that $\mu_i(A) > 1 - \epsilon$ and $\mu_j(A) < \epsilon$ for $j \neq i$.

For every $n \in \mathbb{N}$, using Lemma 3.2 choose mutually disjoint G -equivalent clopen sets

$$A_1^{(n)}, A_2^{(n)}, \dots, A_{2^n}^{(n)} \subset A \text{ such that } \mu_i \left(A \setminus \bigcup_{j=1}^{2^n} A_j^{(n)} \right) < \epsilon.$$

Set

$$c_j^{(n, \epsilon)} = \mu_j \left(\bigcup_{p=1}^{2^n} A_p^{(n)} \right).$$

Observe that $c_i^{(n, \epsilon)} > 1 - 2\epsilon$ and $c_j^{(n, \epsilon)} < \epsilon$ for all $j \neq i$. Choose involutions s_1, \dots, s_{2^n-1} such that

$$s_j(A_j) = A_{j+1} \text{ and } \text{supp}(s_j) = A_j \cup A_{j+1} \text{ for each } 1 \leq j < 2^n.$$

Denote by H_n the subgroup generated by s_1, \dots, s_{2^n-1} . Notice that H_n is isomorphic to the symmetric group $S(2^n)$ on 2^n elements. Denote by $f_{n, \epsilon}$ the isomorphism of $S(2^n)$ onto H_n .

Let $s \in S(2^n)$ and $\text{fix}(s) = \{1 \leq j \leq 2^n : s(j) = j\}$. Since $\mu_j(A_p^{(n)}) = c_j^{(n, \epsilon)} / 2^n$ for each $1 \leq j \leq k, 1 \leq p \leq 2^n$, we get that

$$\mu_j(\text{Fix}(f_{n, \epsilon}(s))) = (1 - c_j^{(n, \epsilon)}) + c_j^{(n, \epsilon)} \frac{\text{card}(\text{fix}(s))}{2^n}, \quad (9)$$

for each $1 \leq j \leq 2^n$ and $s \in S(2^n)$.

Set $\rho_{n,\epsilon}(s) = \chi_\alpha(f_{n,\epsilon}(s))$ for every $s \in S(2^n)$. Then $\rho_{n,\epsilon}$ is a character of the group $S(2^n)$. It follows from the equations (8) and (9) that

$$\rho_n(s) \stackrel{\text{def}}{=} \lim_{\epsilon \rightarrow 0} \rho_{n,\epsilon}(s) = \left(\frac{\text{card}(\text{fix}(s))}{2^n} \right)^{\alpha_i}, \quad s \in S(2^n),$$

is a character of $S(2^n)$. Consider the full group $S(2^\infty)$ of the Bratteli diagram associated to the 2-odometer, see [D] for description of this group. Let ν be the unique invariant measure for $S(2^\infty)$. Consider the function $\rho(s) = \nu(\text{Fix}(s))^{\alpha_i}$, $s \in S(2^\infty)$. Notice that the symmetric group $S(2^n)$ can be seen as a subgroup of $S(2^\infty)$ that permutes only the first n edges of every infinite path. Moreover, it is easy to see that $\rho(s) = \rho_n(s)$ for every $s \in S(2^n)$ and $n \geq 1$. Since for every $n \geq 1$ the function $\rho_n : S(2^n) \rightarrow \mathbb{C}$ is a character, we get that $\rho : S(2^\infty) \rightarrow \mathbb{C}$ is also a character.

Proposition 12 in [D] implies that the function ρ can be a character only if α_i is an integer. We would like to mention that the proof of [D, Proposition 12] is self-contained and is independent of other results from [D]. □

Remark 5.7. (1) The theorem above implies that if $\{A_n\}$ is a sequence of clopen sets such that $\mu(A_n) \rightarrow 0$ for every $\mu \in \mathcal{M}(G)$, then $\text{tr}(P^{A_n}) \rightarrow 1$ as $n \rightarrow \infty$ provided the representation is built over a non-regular character.

(2) If some number $\alpha_i = 0$ in the theorem above, then the trace does not depend on the measure μ_i .

Theorem 5.6 implies that non-regular indecomposable characters on *simple* full groups are automatically continuous with respect to the uniform topology. Observe that this is not true if the group G is not simple. For instance, characters of the form $\chi(g) = \rho([g])$ (see Conjecture in Section 2) are discontinuous if ρ is non-trivial. We would like to mention that the automatic continuity (in a stronger sense) holds for many “big” transformation groups preserving various structures, see, for example, [KeRo], [RS], [R], and [KiT]. Using Proposition 2.23 and Theorem 5.6, we get the following result.

Corollary 5.8. *Let G be the simple full group of a Bratteli diagram B . Let X be the path-space of the diagram B . Assume that the system (X, G) only has a finite number of ergodic measures. Then any finite factor representation of the group G corresponding to a non-regular character is automatically continuous with respect to the uniform topology.*

As a corollary of Theorem 5.6, we also obtain the classification of continuous characters for full groups of minimal dynamical systems. Let $T : X \rightarrow X$ be a homeomorphism of a Cantor set X . Assume that the dynamical system (X, T) is minimal, i.e every orbit of T is dense in X . Denote by $[T]$ the set of all homeomorphisms S of X such that $Sx = T^{n_S(x)}x$, $x \in X$, for some function $n_S : X \rightarrow \mathbb{Z}$. Then $[T]$ is a group, termed the *full group of the system* (X, T) , see [GPS2] and [BM] for more details on this group.

Assume that the system (X, T) has a finitely many of ergodic measures, say μ_1, \dots, μ_k . Endow the group $[T]$ with the uniform topology. Note that the metric generating this topology is defined by

$$D(g, h) = \max_{i=1, \dots, k} \mu_i(\{x \in X : g(x) \neq h(x)\}).$$

Denote by $\pi : [T] \rightarrow \mathcal{M}_\pi$ a continuous factor representation of the full group $[T]$.

Each minimal dynamical system can be represented as a Vershik map acting on the path space of some Bratteli diagram, see [HPS]. Let G be the full group of the Bratteli diagram associated to the system (X, T) . Observe that the group G is dense in $[T]$ with respect to the uniform topology, see [BDK, Theorem 2.8] or [Me, Theorem 2.1] for a more general result. This shows that the group $\pi(G)$ generates the von Neumann algebra \mathcal{M}_π .

Let χ be an indecomposable character of $[T]$ corresponding to the representation π . Since $\pi(G)$ generates \mathcal{M}_π , the restriction of χ onto G is an indecomposable character of G . Using Theorem 5.6, we get the following result describing the structure of continuous characters on the full group $[T]$. In particular, this result gives an algebraic criterion of unique ergodicity for (X, T) .

Corollary 5.9. *Let (X, T) be a Cantor minimal system. Assume that T has only a finitely many ergodic measures μ_1, \dots, μ_k . Let $\chi : [T] \rightarrow \mathbb{C}$ be a continuous (with respect to the uniform topology) indecomposable character. Then there exist parameters $\alpha_1, \dots, \alpha_k \in \{0, 1, \dots, \infty\}$ such that*

$$\chi(g) = \mu_1(\text{Fix}(g))^{\alpha_1} \mu_2(\text{Fix}(g))^{\alpha_2} \cdots \mu_k(\text{Fix}(g))^{\alpha_k}$$

for any $g \in [T]$.

6 Examples of Factor Representations

In this section we construct all possible (up to quasi-equivalence) finite factor representations for the simple full group G under the assumption that the group G has finitely many ergodic measures. The constructions of group representations presented in this section have been known in ergodic theory, see, for example, [FM2]. We show that these representations give rise to indecomposable characters.

6.1 Regular character

We will first consider the case of the regular character χ , i.e. $\chi(g) = 0$ if $g \neq id$ (the identity of the group) and $\chi(id) = 1$. Set $H = l^2(G)$. Define the representation π of G on H by $(\pi(g)f)(h) = f(g^{-1}h)$, $h \in G$. Set $\xi = \delta_{id}$. Clearly, the vectors $\{\pi(g)\xi : g \in G\}$ are mutually orthogonal. Consider the group von Neumann algebra $\mathcal{M}_\pi = \{\pi(g) : g \in G\}''$. Since the group G has the infinite conjugacy class property, the algebra \mathcal{M}_π is a II_1 factor. The unique trace on \mathcal{M}_π is given by $\text{tr}(Q) = (Q\xi, \xi)$, $Q \in \mathcal{M}_\pi$. Therefore, $\chi(g) = \text{tr}(\pi(g))$ is a regular character.

6.2 Non-regular characters

Denote by \tilde{X} the orbit equivalence relation of (X, G) , i.e.

$$\tilde{X} = \{(x, y) \in X \times X : x \sim y\},$$

where $x \sim y$ if there is $g \in G$ such that $g(x) = y$. Note that \tilde{X} is a Borel subset of $X \times X$. Assume that $\{\mu_1, \dots, \mu_k\}$ is a family of ergodic measures for the dynamical system (X, G) . Each ergodic measure μ_i can be lifted up to the *left counting measure* on \tilde{X} by

$$\tilde{\mu}_i(A) = \int_X \text{card}(A_x) d\mu_i(x),$$

where $A \subset \tilde{X}$ is a Borel set and $A_x = \{(x, y) \in A\}$, see [FM1, Theorem 2] for the details.

The group G has the left and right action on the space \tilde{X} defined as $g(x, y) = (gx, y)$ and $g(x, y) = (x, gy)$, $(x, y) \in \tilde{X}$, $g \in G$, respectively. The measure $\tilde{\mu}_i$ is Borel and invariant with respect to both the left and right action of G . Furthermore, the restriction of $\tilde{\mu}_i$ on the diagonal $\Delta = \{(x, x) : x \in X\}$ coincides with the measure μ_i (after the identification of Δ with X).

Consider the Hilbert space $H_i = L^2(\tilde{X}, \tilde{\mu}_i)$. Define the left representation of the group G on H by

$$(\pi_i(g)f)(x, y) = f(g^{-1}x, y), \quad f \in H_i.$$

Similarly, the right representation π'_i of G is defined as $(\pi'_i(g)f)(x, y) = f(x, g^{-1}y)$, $f \in H_i$. The right representation will play an auxiliary and technical role in our construction. Denote the characteristic function of the diagonal $1_\Delta(x, y) \in H_i$ by ξ_i . Set

$$\chi_i(g) = (\pi_i(g)\xi_i, \xi_i) \text{ for } g \in G.$$

Using straightforward computations, one can get that

$$\chi_i(g) = \mu_i(\{x \in X : g(x) = x\}) \text{ for all } g \in G. \quad (10)$$

Then for every $g, h \in G$, we have that

$$\begin{aligned} \chi_i(h^{-1}gh) &= \mu_i(\{x \in X : ghx = hx\}) \\ &= \mu_i(h^{-1}\{y \in X : gy = y\}) \\ &= \chi_i(g). \end{aligned}$$

Notice also that the function χ_i is positive-definite. Hence χ_i is a character.

Fix non-negative integers $\alpha_1, \dots, \alpha_k$ and set

$$H = H_1^{\otimes \alpha_1} \otimes \dots \otimes H_k^{\otimes \alpha_k}, \quad \xi = \xi_1^{\otimes \alpha_1} \otimes \dots \otimes \xi_k^{\otimes \alpha_k}, \quad \chi = \prod \chi_i^{\alpha_i}.$$

Expand the representations π_i and π'_i to unitary representations on H by

$$\pi = \pi_1^{\otimes \alpha_1} \otimes \dots \otimes \pi_k^{\otimes \alpha_k} \text{ and } \pi' = (\pi'_1)^{\otimes \alpha_1} \otimes \dots \otimes (\pi'_k)^{\otimes \alpha_k}.$$

Thus, π is a unitary representation of G on the space H . Denote by \mathcal{M}_π the von Neumann algebra generated by the left representation $\pi(G)$. Set $H_0 = \overline{\text{Lin}(\pi(G)\xi)}$. We can assume that the algebra \mathcal{M}_π is only defined on the Hilbert space H_0 . Hence, ξ is a cyclic vector for \mathcal{M}_π .

We also observe that ξ is separating for the algebra \mathcal{M}_π . Assume the converse, i.e. that there is a non-zero operator $R \in \mathcal{M}_\pi$ such that $R\xi = 0$. Since the representations π and π' commute, we get that $0 = \pi'(g)R\xi = R\pi'(g)\xi$ for every $g \in G$. The identity $\pi(g)\xi = \pi'(g^{-1})\xi$, $g \in G$, implies that $\{\pi'(G)\xi\}$ is dense in H_0 . Hence $R = 0$, which is a contradiction.

Note also that the GNS-construction applied to the character χ yields precisely the representation (H_0, π, ξ) . It follows from the equation (10) that

$$\chi(g) = \mu_1(\text{Fix}(g))^{\alpha_1} \cdots \mu_k(\text{Fix}(g))^{\alpha_k} \text{ for all } g \in G. \quad (11)$$

Our goal is to show that the character χ is indecomposable. This will imply that \mathcal{M}_π is a II_1 -factor. We mention a recent work of Vershik [V2], where he sketches a proof of the fact that the character χ constructed by one measure μ_i and $\alpha_i = 1$ is always indecomposable provided that the group action is totally non-free.

Lemma 6.1. *Let G be the full group of a Bratteli diagram. For any group elements g and h there exists a sequence $\{h_n\}_{n=1}^\infty$ such that $\chi(gh_n) \rightarrow \chi(g)\chi(h)$ for any character χ defined as in the equation (11).*

Proof. Fix a character χ defined by measures as in the equation (11). For each $i = 1, \dots, k$, set $c_i = \mu_i(\text{Fix}(h))$. Using Lemma 5.1, find a sequence of G_n -invariant clopen sets B_n with $\mu_i(B_n) \rightarrow c_i$, $i = 1, \dots, k$, as $n \rightarrow \infty$. Consider the clopen partition

$$\Xi_n = \{C_v^{(i)} : v \in V_n, i = 0, \dots, p_v^{(n)} - 1\}$$

determined by the level n of the diagram. Choose h_n as any element of the group G such that $\text{Fix}(h_n) = B_n$ and $h_n(C_v^{(i)}) = C_v^{(i)}$ for all $v \in V_n$ and $i = 0, \dots, p_v^{(n)} - 1$. It is not hard to see that $\text{supp}(gh_n) = \text{supp}(g) \cup \text{supp}(h_n)$ for every $g \in G_n$. Hence $\text{Fix}(gh_n) = \text{Fix}(g) \cap \text{Fix}(h_n)$. Using Lemma 5.4, we obtain that

$$\begin{aligned} \chi(gh_n) &= \mu_1(\text{Fix}(g) \cap \text{Fix}(h_n))^{\alpha_1} \cdots \mu_k(\text{Fix}(g) \cap \text{Fix}(h_n))^{\alpha_k} \\ &\rightarrow \mu_1(\text{Fix}(g))^{\alpha_1} \cdots \mu_k(\text{Fix}(g))^{\alpha_k} \cdot c_1^{\alpha_1} \cdots c_k^{\alpha_k} \\ &= \chi(g)\chi(h). \end{aligned}$$

□

Theorem 6.2. *Let G be the simple full group of a Bratteli diagram B . Assume that the group G has only finitely many ergodic measures $\{\mu_1, \dots, \mu_k\}$ on the path-space of B . Let χ be a character on G of the form*

$$\chi(g) = \mu_1(\text{Fix}(g))^{\alpha_1} \cdots \mu_k(\text{Fix}(g))^{\alpha_k} \text{ for all } g \in G,$$

where $\alpha_1, \dots, \alpha_k \in \mathbb{N} \cup \{0\}$. Then χ is indecomposable.

Proof. If the character χ is decomposable, then there exists a standard probability measure space (Γ, ν) such that

$$\chi(g) = \int_{\Gamma} \chi_{\gamma}(g) d\nu(\gamma),$$

where χ_{γ} is an indecomposable character, see, for example, [Ta, Chapter IV, Theorem 8.21]. In view of Theorem 5.6, there is at most a countable number of indecomposable characters. Hence

$$\chi(g) = \sum_{n \geq 1} \alpha_n \chi_n(g),$$

where $\alpha_n > 0$ and χ_n is an indecomposable character built as in the equation (8). We notice that the regular character does not show up amongst $\{\chi_n\}_{n \geq 1}$. Observe also that $1 = \chi(id) = \sum_{n \geq 1} \alpha_n$.

Fix any two elements $a, b \in G$. Choose a sequence $\{a_n\}$ as in Lemma 6.1. It follows from Lemma 6.1 and the dominated convergence theorem that

$$\chi(a)\chi(b) = \lim_{n \rightarrow \infty} \chi(ab_n) = \lim_{n \rightarrow \infty} \sum_{i \geq 1} \alpha_i \chi_i(ab_n) = \sum_{i \geq 1} \alpha_i \chi_i(a) \chi_i(b).$$

Consider $g, h \in G$. Applying the Cauchy-Schwarz inequality, we get that

$$\begin{aligned} \chi(g)\chi(h) &= \sum_{i \geq 1} \alpha_i \chi_i(g) \chi_i(h) \\ &\leq \left(\sum_{i \geq 1} \alpha_i \chi_i(g) \chi_i(g) \right)^{1/2} \cdot \left(\sum_{i \geq 1} \alpha_i \chi_i(h) \chi_i(h) \right)^{1/2} \\ &= (\chi(g)\chi(g))^{1/2} (\chi(h)\chi(h))^{1/2} \\ &= \chi(g)\chi(h). \end{aligned}$$

The equality here means that $\chi_i(g) = c \chi_i(h)$ for all indices $i \geq 1$, where c is some constant. Hence $\chi(g) = c \chi(h)$. Repeating these arguments for $h = id$, we get that $\chi_i(g) = q$ for all $i \geq 1$. Thus, $\chi(g) = q = \chi_i(g)$ for all $i \geq 1$. Since the element g was chosen arbitrarily, we get that $\chi \equiv \chi_i$ for all i , i.e. χ is indecomposable. \square

It follows from Theorem 5.6 that the list of characters constructed in this section is exhaustive.

7 Rational Permutations of the Unit Interval

In the paper [GoPe] (see also [Go]) Goryachko and Petrov described all indecomposable characters on the group R of rational permutations of the unit interval $[0, 1)$. In this section we will apply Theorem 5.6 to get a new proof of their result.

Denote by $S(n)$ the group of permutation on the set $\{0, 1, \dots, n-1\}$. For any $n \in \mathbb{N}$ and a permutation $s \in S(n)$, define a bijection $g_s : [0, 1) \rightarrow [0, 1)$, which permutes subintervals $[0, 1/n), \dots, [1 - 1/n, 1)$ as follows

$$g_s(x) = \frac{s([nx]) + \{nx\}}{n},$$

where $[a]$ and $\{a\}$ is the integer and the fraction part of the number a , respectively.

The group R is defined as the set of all bijections $\{g_s : s \in S(n), n \in \mathbb{N}\}$. The detailed structure of the group R is described in [Go], [GoPe]. The representation of R as an inductive limit (presented below) is taken from [Go].

For each $n, m \in \mathbb{N}$, define an embedding $S(n) \rightarrow S(nm)$ as

$$s \rightarrow s', \quad s'(j) = ms([j/m]) + (j \bmod m), \quad s \in S(n), \quad s' \in S(nm). \quad (12)$$

These embeddings are called *periodic*. Observe that the map $s \rightarrow g_s, s \in S(n)$ and $g_s \in R$, is invariant under the periodic embeddings. We can introduce a partial ordering on \mathbb{N} by saying that n is less than m if n divides m . The family of symmetric groups $\{S(n)\}_{n \in \mathbb{N}}$ is a directed family with respect to this partial ordering. The group R is an inductive limit of this family with respect to the periodic embeddings, see [Go, Proposition 2].

Observe that each group $S(n)$ is embedded by a periodical embedding into a group $S(n!)$. This shows that R is the inductive limit of the sequence of groups $\{S(n!)\}_{n \geq 1}$ with respect to the periodic embeddings.

Let $B_R = (V, E)$ be a diagram such that (i) on each level $n \geq 0$, the diagram has exactly $n+1$ vertices and (ii) each vertex from V_n is connected to each vertex of V_{n+1} , $n \geq 0$, exactly by one edge. In particular, $\text{card}(E_n) = n(n+1)$. The following figure gives a diagrammatic representation of B_R .

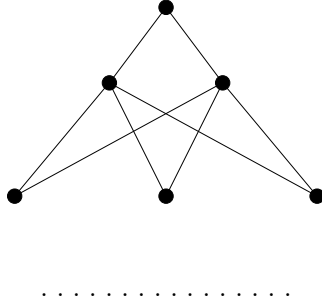


Fig. 1

Proposition 7.1. *The full group G of the Bratteli diagram B_R is isomorphic to the group of rational permutations R .*

Proof. For each $n \geq 1$, enumerate vertices from V_n by the numbers $0, 1, \dots, n$. Then each finite path p starting at v_0 and terminating at a vertex from V_n can be uniquely encoded by a sequence of numbers (a_1, \dots, a_n) , where the entry $a_j \in \{0, 1, \dots, j\}$ represents the vertex from V_j the path p goes through.

Denote by X_n the set of all finite paths terminating at the level n . Let G_n be the subgroup of the full group G consisting of homeomorphisms that only act on X_n . Define a bijection $i_n : X_n \rightarrow \{0, 1, \dots, (n+1)! - 1\}$ by

$$i_n(a_1, \dots, a_n) = a_1 \frac{(n+1)!}{2!} + a_2 \frac{(n+1)!}{3!} + \dots + a_n.$$

The bijection i_n induces an embedding φ_n of G_n into $S((n+1)!)$. The image $\varphi_n(G_n)$ consists of all permutations that do not change the remainder modulo $n+1$. It follows that $\varphi_n(G_n)$ contains the image of $S(n!)$ in $S((n+1)!)$ under the periodic embedding. Observe that the embeddings φ_n commute with the embeddings $S(n!) \rightarrow S((n+1)!)$ and $G_n \rightarrow G_{n+1}$. This implies that the inductive limit of the groups G_n is isomorphic to the inductive limit of the groups $S(n!)$. \square

For each pair of vertices $(v, w) \in V_n \times V_m$, $n < m$, denote by $f_{v,w}^{(n,m)}$ the number of finite paths connecting v to w . Similarly, let $f_w^{(m)}$ stand for the number of paths from the top vertex v_0 to the vertex w . Let λ be a G -invariant ergodic measure on the diagram B_R . By the pointwise ergodic theorem there exists an infinite path x such that the measure of any cylinder set $U = U(e_1, \dots, e_n)$ can be found by

$$\lambda(U) = \lim_{m \rightarrow \infty} f_{v,w_m}^{(n,m)} / f_{w_m}^{(m)},$$

where w_m is the vertex of level m the path x goes through and v is the vertex of level n the cylinder set U terminates at, see the details in [VK2, Theorem 2] or [BKMS1, Proposition 5.1]. Due to the structure of the diagram B_R , the quantity $f_{v,w}^{(n,m)} / f_w^{(m)}$ does not depend on the choice of vertices v and w . Therefore, λ is the only G -invariant measure. Thus, the following result is an immediate corollary of Theorem 5.6.

Corollary 7.2 (Goryachko and Petrov). *The functions $\chi_k(g) = \lambda(\text{Fix}(g))^k$, $k \in \mathbb{N} \cup \{0, \infty\}$, $g \in R$, are the only indecomposable characters of the group R of rational permutations of the unit interval.*

Remark 7.3. Note that χ_0 and χ_∞ in the corollary above are the identity character and the regular character, respectively.

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References

- [BDK] S. Bezuglyi, A.H. Dooley, J. Kwiatkowski, *Topologies on the group of Borel automorphisms of a standard Borel space*. Topol. Methods in Nonlinear Analysis, **27** (2006), 333 – 385.
- [BKMS1] S. Bezuglyi, J. Kwiatkowski, K. Medynets, and B. Solomyak, *Invariant measures on stationary Bratteli diagrams*. Ergodic Theory Dynam. Systems, **30** (2010), no. 4, 973 – 1007.
- [BKMS2] S. Bezuglyi, J. Kwiatkowski, K. Medynets, and B. Solomyak, *Finite rank Bratteli diagrams: Structure of invariant measures*. Preprint (2011), arXiv:1003.2816.
- [BM] S. Bezuglyi and K. Medynets, *Full groups, flip conjugacy, and orbit equivalence of Cantor minimal systems*. Colloq. Math. **110** (2008), no. 2, 409 – 429.
- [BR] O. Brattelli and W. Robinson, *Operator algebras and quantum statistical mechanics. C^* - and W^* -algebras. Symmetry Groups. Decomposition of States*. Texts and Monographs in Physics. Berlin Heidelberg-New York, Springer-Verlag, 1987.
- [D] A. Dudko, *Characters on the full group of the odometer*. to appear in Journal of Functional Analysis (2011).
- [FM1] J. Feldman and C. Moore, *Ergodic Equivalence Relations, Cohomology, and Von Neumann Algebras. I*. Trans. of the AMS, **234**, No. 2. (1977), 289 – 324.
- [FM2] J. Feldman and C. Moore, *Ergodic Equivalence Relations, Cohomology, and Von Neumann Algebras. II*. Trans. of the AMS, **234**, No. 2. (1977), 325 – 359.
- [FFT] S. Ferenczi, A. Fisher, M. Talet, *Minimality and unique ergodicity for adic transformations*. J. Anal. Math. **109** (2009), 1 – 31.
- [GPS1] T. Giordano, I. Putnam, and C. Skau, *Topological orbit equivalence and C^* -crossed products*. J. Reine Angew. Math., **469** (1995), 51 – 111.
- [GPS2] T. Giordano, I. Putnam, and C. Skau, *Full groups of Cantor minimal systems*. Israel J. Math. **111** (1999), 285 – 320.
- [GW] E. Glasner, B. Weiss, *Weak orbit equivalence of Cantor minimal systems*. Internat. J. Math., **6** (1995), 559 – 579.
- [GK] R. Gohm and C. Köstler, *Non-commutative independence from characters of the infinite symmetric groups S_∞* . Preprint (2010), arXiv:1005.5726.

- [Go] E. Goryachko, *The K_0 -functor and the characters of the group of rational rearrangements of a segment*. (Russian) Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **360** (2008), Teoriya Predstavlenii, Dinamicheskie Sistemy, Kombinatornye Metody. XVI, 124 – 138, 297.
- [GoPe] E. Goryachko and F. Petrov, *Indecomposable characters of the group of rational rearrangements of a segment*. (Russian) Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **378** (2010), Teoriya Predstavlenii, Dinamicheskie Sistemy, Kombinatornye Metody. XVIII, 17 – 31, 228.
- [Gr] R. Grigorchik, *Some topics of dynamics of group actions on rooted trees*. The Proceedings of the Steklov Institute of Math., v. **273** (2011), 1 – 118.
- [H] P. Halmos, *Lectures on ergodic theory*. Chelsea Publishing Co., New York, 1960.
- [HPS] R. H. Herman, I. Putnam, and C. Skau, *Ordered Bratteli diagrams, dimension groups, and topological dynamics*. Internat. J. Math., **3** (1992), 827 – 864.
- [KR1] R. Kadison and J. Ringrose, *Fundamentals of the theory of operator algebras. Vol. I. Elementary theory*. Pure and Applied Mathematics, 100. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York, 1983.
- [KR2] R. Kadison and J. Ringrose, *Fundamentals of the theory of operator algebras. Vol. II. Advanced theory*. Pure and Applied Mathematics, 100. Academic Press, Inc., Orlando, FL, 1986.
- [KeRo] A. Kechris and C. Rosendal, *Turbulence, amalgamation, and generic automorphisms of homogeneous structures*. Proc. Lond. Math. Soc. (3) **94** (2007), no. 2, 302 – 350.
- [K] S. Kerov, *Asymptotic representation theory of the symmetric group and its applications in analysis*. Translations of Mathematical Monographs, 219. American Mathematical Society, Providence, RI, 2003.
- [KV] S. Kerov and A. Vershik, *Locally semisimple algebras. Combinatorial theory and the K_0 -functor*. (Russian) Current problems in mathematics. Newest results, Vol. **26**, 3 – 56. Itogi Nauki i Tekhniki, Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, 1985.
- [KiT] J. Kittrell and T. Tsankov, *Topological properties of full groups*. Ergodic Theory Dynam. Systems **30** (2010), no. 2, 525 – 545.

- [KrSu] N. Kroshko and V. Sushchansky, *Direct limits of symmetric and alternating groups with strictly diagonal embeddings*. Arch. Math. **71** (1998), 173 – 182.
- [LN] Ya. Lavrenyuk and V. Nekrashevych, *On classification of inductive limits of direct products of alternating groups*. J. Lond. Math. Soc. (2) **75** (2007), no. 1, 146 – 162.
- [LS] Ya. Lavrenyuk and V. Sushchanskii, *Locally finite groups associated with Bratteli diagrams*. (Ukrainian) Mat. Stud. **31** (2009), no. 1, 3 – 11.
- [LePu] F. Leinen and O. Puglisi, *Positive definite functions of diagonal limits of finite alternating groups*. J. London Math. Soc. (2) **70** (2004), no. 3, 678 – 690.
- [M] H. Matui, *Some remarks on topological full groups of Cantor minimal systems*. Internat. J. Math. **17** (2006), no. 2, 231 – 251.
- [Me] K. Medynets, *On approximation of homeomorphisms of a Cantor set*. Fundamenta Mathematicae **194** (2007), no. 1, 1 – 13.
- [Mel] X. Méla, *A class of nonstationary adic transformations*. Ann. Inst. H. Poincaré Probab. Statist. **42** (2006), no. 1, 103 – 123.
- [PV] K. Petersen and A. Varchenko, *The Euler adic dynamical system and path counts in the Euler graph*. Tokyo J. Math. **33** (2010), no. 2, 327 – 340.
- [Ok] A. Okounkov, *On the representation of the infinite symmetric group*. Representation theory, dynamical systems, combinatorial and algorithmic methods. Part II, Zap. Nauchn. Sem. POMI, **240**, POMI, St. Petersburg, 1997, 166 – 228.
- [R] C. Rosendal, *Automatic continuity of group homomorphisms*. Bull. Symbolic Logic **15** (2009), no. 2, 184214.
- [RS] C. Rosendal S. Solecki, *Automatic continuity of homomorphisms and fixed points on metric compacta*. Israel J. Math. **162** (2007), 349 – 371.
- [Ta] M. Takesaki, *Theory of operator algebras I*. Encyclopedia of Mathematical Sciences, vol. 124.
- [Th] E. Thoma, *Die unzerlegbaren, positiv-definiten Klassenfunktionen der abzählbar unendlichen, symmetrischen Gruppe*. (German) Math. Z. **85** (1964), 40 – 61.
- [VK1] A.M. Vershik and S.V. Kerov, *Characters and factor representations of the infinite symmetric group*. (Russian) Dokl. Akad. Nauk SSSR **257** (1981), no. 5, 1037 – 1040.

- [VK2] A.M. Vershik and S.V. Kerov. *Asymptotic theory of the characters of a symmetric group*. (Russian) Funktsional. Anal. i Prilozhen., **15**(4) (1981), 15 – 27.
- [V1] A. Vershik, *Two lectures on the asymptotic representation theory and statistics of Young diagrams. Asymptotic combinatorics with applications to mathematical physics (St. Petersburg, 2001)*, 161 – 182, Lecture Notes in Math., **1815**, Springer, Berlin, 2003.
- [V2] A. Vershik, *Nonfree Actions of Countable Groups and their Characters*. Nonfree actions of countable groups and their characters. (Russian) Zapiski Nauchn. Semin. POMI **378**, 5 – 16 (2010). English translation to appear in J. Math. Sci.
- [Z] A.E. Zalesski, *Group Rings of Simple Locally Finite Groups*. "Finite and Locally Finite Groups", Proceedings of the NATO Advanced Study Institute, Istanbul, Turkey, 14–27 August 1994. NATO Science Series C, Vol. 471, (eds. B. Hartley. G.M.. Seitz, A.V.. Borovik, R.M.. Bryant, Kluwer, Dordrecht, 1995).